$$\vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longrightarrow \boxed{\vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

$\vec{v_1}, \vec{v_2}, \vec{v_3}$ sono linearmente indipendenti!

$$c_1 \vec{v_1} + c_2 \vec{v_2} + c_3 \vec{v_3} = 0$$

$$c_1 \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \iff \begin{cases} c_1 = 0 \\ c_2 = 0 \\ -2c_1 - c_2 + c_3 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$

Esiste la matrice $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix}$ per cui A è diagonalizzabile.

Siano $a, \alpha, \beta \in \mathbb{R}$ e consideriamo la matrice $A = \begin{pmatrix} \alpha & a \\ 0 & \beta \end{pmatrix}$ dimostrare che se $\alpha = \beta$ allora A non è diagonalizzabile.

$$\det \begin{pmatrix} \alpha - \lambda & a \\ 0 & \beta - \lambda \end{pmatrix} = 0 \qquad (\alpha - \lambda)(\beta - \lambda) = 0 \qquad \lambda_{1,2} = \alpha, \beta$$

$$A\vec{v_1} = \lambda_1 \vec{v_1} \Rightarrow \begin{cases} \alpha x + a y = \alpha x \\ \beta y = \alpha y \end{cases} \qquad \begin{cases} y = 0 \\ 0 = 0 \end{cases}$$

$$\vec{v_1} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \boxed{\vec{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$A\vec{v_2} = \lambda_2 \vec{v_2} \Rightarrow \begin{cases} \alpha x + a y = \beta x \\ \beta y = \beta y \end{cases} \qquad \begin{cases} x = \frac{a}{\beta - \alpha} y \quad \text{se } \beta \neq \alpha \\ y \text{ arbitrario} \end{cases}$$

$$\vec{v_2} = \begin{pmatrix} \frac{a}{\beta - \alpha} y \\ y \end{pmatrix} = y \begin{pmatrix} \frac{a}{\beta - \alpha} \\ 1 \end{pmatrix} \longrightarrow \boxed{\vec{v_2} = \begin{pmatrix} \frac{a}{\beta - \alpha} \\ 1 \end{pmatrix}}$$

Dunque se $\alpha \neq \beta$ $\begin{pmatrix} \alpha & a \\ 0 & \beta \end{pmatrix}$ è diagonalizzabile perché

esistono più autovettori $\vec{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ e $\vec{v_2} = \begin{pmatrix} \frac{a}{\beta-\alpha} \\ 1 \end{pmatrix}$.

Se $\alpha = \beta$ allora $A\vec{v_2} = \lambda_2 \vec{v_2} \Rightarrow \begin{cases} ay = 0 \\ \beta y = \beta y \end{cases} \to y = 0$

$$\vec{v_2} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \vec{v_2} = \vec{v_1}.$$

esiste un solo autovettore.

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \qquad \det \begin{pmatrix} 1-\lambda & 1 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)^2 + 1\right] = 0 \qquad (1-\lambda)(1 - 2\lambda + \lambda^2 + 1) = 0$$

$$(1-\lambda)(\lambda^2 - 2\lambda + 2) = 0 \qquad \lambda^2 - 2\lambda + 2 = 0 \qquad \lambda = \frac{1 \pm \sqrt{1-2}}{1} = 1 \pm i$$

$\lambda_1 = 1+i$, $\lambda_2 = 1-i$, $\lambda_3 = 1$.

A non è diagonalizzabile nel campo reale. Ovviamente per questi valori è possibile determinare degli autovettori ma questi non avranno componenti reali.

Consideriamo la matrice delle rotazioni

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & +\cos\theta \end{pmatrix} \qquad \det \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & +\cos\theta - \lambda \end{pmatrix} = 0$$

$$+(\cos\theta - \lambda)^2 \cancel{(\cos\theta + \lambda)} + \sin^2\theta = 0 \qquad \lambda^2 - 2\cos\theta\,\lambda + \cos^2\theta + \sin^2\theta = 0$$

$$\lambda^2 - 2\cos\theta\,\lambda + 1 = 0 \qquad \lambda_{1,2} = \cos\theta \pm \sqrt{\cos^2\theta - 1} =$$

$$= \cos\theta \pm \sqrt{-\sin^2\theta} = \cos\theta \pm i\sin\theta = e^{\pm i\theta};$$

$$\boxed{\lambda_1 = e^{i\theta} \; ; \; \lambda_2 = e^{-i\theta}};$$

Gli autovettori sono complessi

$$A\vec{v}_1 = \lambda_1 \vec{v}_1 \Rightarrow \begin{cases} \cos\theta\, x - \sin\theta\, y = e^{i\theta} x \\ \sin\theta\, x + \cos\theta\, y = e^{i\theta} y \end{cases}$$

$$\begin{cases} y = \dfrac{\cos\theta - e^{i\theta}}{\sin\theta} x \\ \text{———} \end{cases} \qquad \begin{cases} y = \dfrac{\cancel{\cos\theta} - \cancel{\cos\theta} - i\cancel{\sin\theta}}{\cancel{\sin\theta}} x = -ix \\ \text{———} \end{cases}$$

$$\vec{v}_1 = \begin{pmatrix} x \\ -ix \end{pmatrix} = x \begin{pmatrix} 1 \\ -i \end{pmatrix} \longrightarrow \boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}}$$

$$A\vec{v}_2 = \lambda_2 \vec{v}_2 \Rightarrow \begin{cases} \cos\theta\, x - \sin\theta\, y = e^{-i\theta} x \\ \sin\theta\, x + \cos\theta\, y = e^{-i\theta} y \end{cases}$$

$$\begin{cases} y = \dfrac{\cos\theta - e^{-i\theta}}{\sin\theta} x \\ \text{———} \end{cases} \qquad \begin{cases} y = \dfrac{\cancel{\cos\theta} - \cancel{\cos\theta} + i\cancel{\sin\theta}}{\cancel{\sin\theta}} x = ix \\ \text{———} \end{cases}$$

$$\vec{v}_2 = \begin{pmatrix} x \\ ix \end{pmatrix} = x \begin{pmatrix} 1 \\ i \end{pmatrix} \longrightarrow \boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}}$$

24

$$A = \begin{pmatrix} 2 & 4 & 0 \\ 1 & 0 & -1 \\ -2 & 0 & 2 \end{pmatrix} \qquad \det \begin{pmatrix} 2-\lambda & 4 & 0 \\ 1 & -\lambda & -1 \\ -2 & 0 & 2-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)\left[-\lambda(2-\lambda)\right] - 4\left[(2-\lambda) - 2\right] = 0$$

$$(2-\lambda)(-2\lambda + \lambda^2) - 4(\cancel{2} - \lambda - \cancel{2}) = 0$$

$$\cancel{-4\lambda} + 2\lambda^2 + 2\lambda^2 - \lambda^3 \cancel{+4\lambda} = 0$$

$$\lambda^3 - 4\lambda^2 = 0 \qquad \lambda^2(\lambda - 4) = 0$$

$$\boxed{\begin{matrix} \lambda_1 = 0 & m = 2 \\ \lambda_2 = 4 & m = 1 \end{matrix}}$$

$$A\vec{v_1} = \lambda_1 \vec{v_1} \Rightarrow \begin{cases} 2x + 4y = 0 \\ x - z = 0 \\ -2x + 2z = 0 \end{cases} \qquad \begin{cases} y = -x/2 \\ z = x \end{cases}$$

$$\vec{v_1} = \begin{pmatrix} x \\ -x/2 \\ x \end{pmatrix} = x \begin{pmatrix} 1 \\ -1/2 \\ 1 \end{pmatrix} \longrightarrow \boxed{\vec{v_1} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}}$$

$$A\vec{v_2} = \lambda_2 \vec{v_2} \Rightarrow \begin{cases} 2x + 4y = 4x \\ x - z = 4y \\ -2x + 2z = 4z \end{cases} \qquad \begin{cases} 2x - 4y = 0 \\ x - 4y - z = 0 \\ 2z + 2x = 0 \end{cases}$$

$$\begin{cases} y = x/2 \\ z = -x \end{cases} \qquad \vec{v_2} = \begin{pmatrix} x \\ x/2 \\ -x \end{pmatrix} = x \begin{pmatrix} 1 \\ 1/2 \\ -1 \end{pmatrix} \longrightarrow \boxed{\vec{v_2} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}}$$

la matrice non è diagonalizzabile