$$\int \frac{x+3}{(x+1)(x^2+9)}\,dx = (*) \qquad \frac{x+3}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

$$\frac{A(x^2+9)+(Bx+C)(x+1)}{(x+1)(x^2+9)} = \frac{(A+B)x^2+(B+C)x+C+9A}{(x+1)(x^2+9)}$$

$$\begin{cases} A+B=0 \\ B+C=1 \\ C+9A=3 \end{cases} \Rightarrow \begin{cases} A=1/5 \\ B=-1/5 \\ C=6/5 \end{cases}$$

$$(*) = \int \frac{1}{5}\,\frac{1}{x+1}\,dx + \int \frac{-1/5\,x + 6/5}{x^2+9}\,dx =$$

$$(*) = \frac{1}{5}\ln|x+1| - \frac{1}{5}\int \frac{x-6}{x^2+9}\,dx = (**)$$

$$\frac{x-6}{x^2+9} = \frac{1}{2}\,\frac{2x-12}{x^2+9} = \frac{1}{2}\,\frac{2x}{x^2+9} - \frac{1}{2}\,\frac{12}{x^2+9} =$$

$$= \frac{1}{2}\,\frac{2x}{x^2+9} - 6\,\frac{1}{9\left((x/3)^2+1\right)} = \frac{1}{2}\,\frac{2x}{x^2+9} - \frac{2}{3}\,\frac{1}{(x/3)^2+1},$$

$$(**) = \frac{1}{5}\ln|x+1| - \frac{1}{5}\left\{\frac{1}{2}\int \frac{2x}{x^2+9}\,dx - \frac{2}{3}\int \frac{1}{(x/3)^2+1}\,dx\right\} =$$

$$= \frac{1}{5}\ln|x+1| - \frac{1}{10}\ln(x^2+9) + \frac{2}{15}\int \frac{1}{y^2+1}\,3\,dy = \quad \text{N.B. } y = x/3$$

$$= \frac{1}{5}\ln|x+1| - \frac{1}{10}\ln(x^2+9) + \frac{2}{5}\,\mathrm{arctg}\,y =$$

$$= \frac{1}{5}\ln|x+1| - \frac{1}{10}\ln(x^2+9) + \frac{2}{5}\,\mathrm{arctg}\,x/3\,;$$

$$\int \frac{dx}{x^6-3x^3+1} \qquad x^6-3x^3+1=0$$

(2)

$$x^3 = \frac{3\pm\sqrt{9-4}}{2} = \frac{3\pm\sqrt{5}}{2}$$

$$x^6-3x^3+1 = \left(x^3-\frac{3-\sqrt5}{2}\right)\left(x^3-\frac{3+\sqrt5}{2}\right) = (x^3-a^3)(x^3-b^3)$$

dove $a=\left(\frac{3-\sqrt5}{2}\right)^{1/3}$, $b=\left(\frac{3+\sqrt5}{2}\right)^{1/3}$

$$(x^3-a^3)(x^3-b^3) = (x-a)(x^2+ax+a^2)(x-b)(x^2+bx+b^2) =$$

$$= (x-a)(x-b)(x^2+ax+a^2)(x^2+bx+b^2)$$

$$x^2+ax+a^2=0 \qquad x=\frac{-a\pm\sqrt{a^2-4a^2}}{2}=\frac{-a\pm\sqrt{-3a^2}}{2}$$

non esistono radici $\Delta<0$ (devono osservare la forma arctg)

Quindi

$$\frac{1}{x^6-3x^3+1} = \frac{1}{(x-a)(x-b)(x^2+ax+a^2)(x^2+bx+b^2)} =$$

$$= \frac{A}{x-a}+\frac{B}{x-b}+\frac{Cx+D}{x^2+ax+a^2}+\frac{Ex+F}{x^2+bx+b^2}$$

Rifacendo i calcoli simili, ed procedendo oltre,

$$\begin{cases} A+B+C+E=0 \\ aA-aC+D-bE+F+bB=0 \\ a^2A+b^2B-aD-bF=0 \\ -Ab^3-a^3B-b^3C-a^3E=0 \\ -ab^3A-a^3bB+ab^3C-b^3D+a^3bE-a^3F=0 \\ -a^2b^3A-a^3b^2B+ab^3D+a^3bF=1 \end{cases}$$

$$A = \frac{1}{3a^2(a^3-b^3)}\,; \quad B = \frac{1}{3b^2(b^3-a^3)}\,;$$

(3)

$$C = \frac{1}{3a^2(b^3-a^3)}\,, \quad D = \frac{2}{3a(b-a)(a^2+ab+b^2)}\,;$$

$$E = \frac{1}{3b^2(b^3-a^3)}\,, \quad F = \frac{2}{3b(a^3-b^3)}\,;$$

N.B. $a^3-b^3 = \frac{3-\sqrt{5}}{2} - \frac{3+\sqrt{5}}{2} = -\sqrt{5}$

$$a^2 = \left(\frac{3-\sqrt{5}}{2}\right)^{2/3}, \quad b^2 = \left(\frac{3+\sqrt{5}}{2}\right)^{2/3}$$

$a^2+ab+b^2 = \ldots$ tocca farlo numer.

Poiché x^2+ax+a^2 e x^2+bx+b^2 hanno $\Delta<0$ dobbiamo costruire come segue:

$$\frac{Cx+D}{x^2+ax+a^2} =$$

$$= C\,\frac{x+D/C}{x^2+ax+a^2} = \frac{C}{2}\,\frac{2x+2D/C}{x^2+ax+a^2} =$$

$$= \frac{C}{2}\,\frac{2x+a-a+2D/C}{x^2+ax+a^2} = \frac{C}{2}\,\frac{2x+a}{x^2+ax+a^2} +$$

$$+ \frac{C}{2}\left(-a+2D/C\right)\frac{1}{x^2+ax+a^2}$$

$$= \frac{e}{2}\frac{d}{dx}\ln\left|x^2+ex+e^2\right| + \frac{e}{2}\left(-a+2D/c\right)\frac{1}{\underbrace{x^2+ex+e^2}_{\sqcup}}$$

$$\sqcup = \frac{1}{\left(x+\frac{a}{2}\right)^2+\frac{3}{4}a^2} = \frac{1}{3/4\,a^2}\,\frac{1}{\left(\frac{x+a/2}{\sqrt{3/4\,e^2}}\right)^2+1} =$$

$$= \frac{4}{3e^2}\,\frac{1}{\left(2\frac{x+a/2}{\sqrt{3}\,a}\right)^2+1},$$

In fine:

$$\boxed{\frac{cx+D}{x^2+ex+e^2} = \frac{e}{2}\frac{d}{dx}\ln\left|x^2+ex+e^2\right| + \frac{e}{2}\left(-a+\frac{2D}{c}\right)\cdot\frac{4}{3e^2}\,\frac{1}{\left(2\frac{x+e/2}{\sqrt{3}\,a}\right)^2+1}}$$

Lo stesso deve valere per $\frac{Ex+F}{x^2+bx+b^2}$.

Dunque l'integrale completo è:

$$\int\frac{dx}{x^6-3x^3+1} = A\int\frac{dx}{x-a} + B\int\frac{dx}{x-b} + \underline{\frac{e}{2}\int dx\,\frac{d}{dx}\ln\left|x^2+ex+e^2\right|}$$

$$\underline{+\frac{e}{2}\left(-a+\frac{2D}{c}\right)\frac{4}{3e^2}\int\frac{1}{\left(2\frac{x+e/2}{\sqrt{3}\,a}\right)^2+1}\,dx} + \left[\boxed{a\to b \quad D\to F \quad c\to E}\right] =$$

$$= A \ln|x-a| + B \ln|x-b| + \frac{C}{2} \ln|x^2+ax+a^2| +$$

$$+ \frac{C}{2}\left(-a+\frac{2D}{C}\right)\frac{4}{3a^2}\int \frac{1}{y^2+1}\,\frac{\sqrt{3}a}{2}\,dy + \frac{E}{2}\ln|x^2+bx+b^2| +$$

$$+ \frac{E}{2}\left(-b+\frac{2F}{E}\right)\frac{4}{3b^2}\int \frac{1}{y^2+1}\,\frac{\sqrt{3}b}{2}\,dy =$$

Abbiamo posto $2\,\dfrac{x+a/2}{\sqrt{3}a} = y \Rightarrow dx = \dfrac{\sqrt{3}a}{2}\,dy$

$$= A \ln|x-a| + B \ln|x-b| + \frac{C}{2}\ln|x^2+ax+a^2| +$$

$$+ \frac{E}{2}\ln|x^2+bx+b^2| + \frac{C}{2}\left(-a+\frac{2D}{C}\right)\frac{4}{3a^2}\cdot\frac{\sqrt{3}a}{2}\cdot$$

$$\cdot \operatorname{arctg}\left(\frac{2x+a}{\sqrt{3}a}\right) + \frac{E}{2}\left(-b+\frac{2F}{E}\right)\frac{4}{3b^2}\,\frac{\sqrt{3}b}{2}\cdot$$

$$\cdot \operatorname{arctg}\left(\frac{2x+b}{\sqrt{3}b}\right).$$

[illegible]

[illegible]