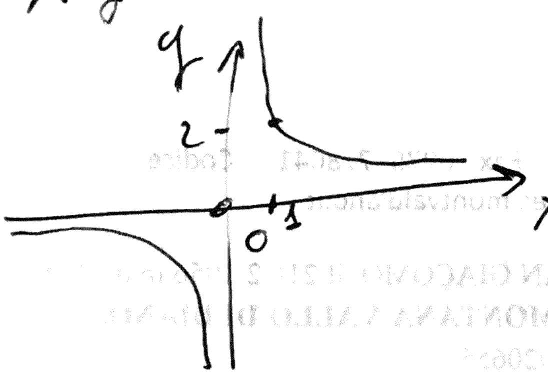
# SVOLGIMENTO PROVA SCRITTA DI GEOMETRIA E ALGEBRA LINEARE

21/1/2019

I

1) $xy = k$

$2 \cdot 1 = k \Rightarrow \boxed{yx = 2}$




$$\begin{cases} x = x_0 + x' \\ y = y_0 + y' \end{cases} \qquad \begin{cases} x = 1 + x' \\ y = -1 + y' \end{cases}$$

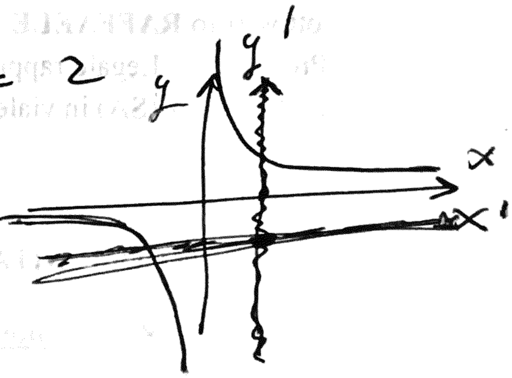
$P \equiv (1, -1)$

$(1+x')(y'-1) = 2$

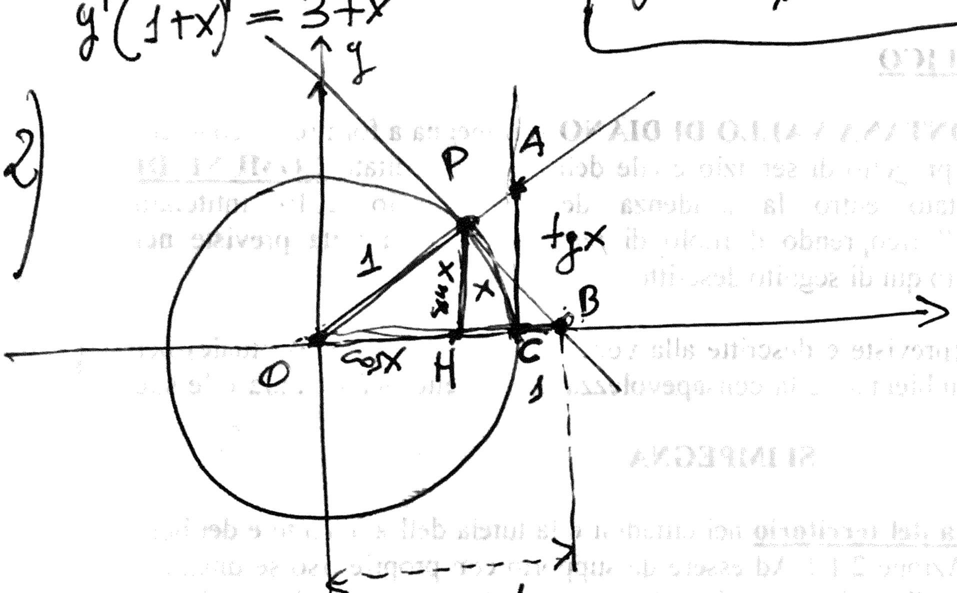
$y' - 1 + x'y' - x' = 2$

$y'(1+x') = 3 + x'$

$$\boxed{y' = \frac{x'+3}{x'+1}}$$




2)




$\sec x = \frac{1}{\cos x}$

$\overline{OB} : \overline{OP} = \overline{OP} : \overline{OH}$

$\frac{\sec x}{1} = \frac{1}{\cos x}$

$\sec x = \frac{1}{\cos x}$

3) $\vartheta_a^x = \arccos\left\{\frac{\vec{a} \cdot \hat{\imath}}{|\vec{a}|}\right\} = \arccos \frac{1}{\sqrt{6}} = \arccos \frac{\sqrt{6}}{6} = 65,91°$ (1,15 rad)

$\vartheta_a^z = \arccos\left(\frac{\vec{a} \cdot \hat{k}}{|\vec{a}|}\right) = \arccos \frac{2}{\sqrt{6}} = \arccos \frac{\sqrt{6}}{3} = 35,26°$ (0,61 rad)

$\vartheta_b^x = \arccos\left(\frac{\vec{b} \cdot \hat{\imath}}{|\vec{b}|}\right) = \arccos \frac{-1}{\sqrt{3}} = \arccos\left(-\frac{\sqrt{3}}{3}\right) = 125,26 \rightarrow 2,19$ rad

$\vartheta_b^z = \arccos\left(\frac{\vec{b} \cdot \hat{k}}{|\vec{b}|}\right) = \arccos \frac{1}{\sqrt{3}} = \arccos \frac{\sqrt{3}}{3} = 54,73°$ (0,96 rad)

$\vec{a} \times \vec{b} = (-1, -3, 2)$

$\vec{a} \cdot \vec{b} = 2$

$\theta = \arccos\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right) = 1,08$ rad $61,87°$

$\theta = \arcsin\left(\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}\right) = 1,08$ rad

II

4)

$\overline{a}_1=(1,0,1,-1)$ $\quad \overline{a}_2=(-1,2,0,1)$

$\overline{a}_3=(0,1,2,-1)$ $\quad \overline{a}_4=(0,3,3,-1)$

$$\det\begin{pmatrix}1&0&1&-1\\0&1&2&-1\\-1&2&0&1\\0&3&3&-1\end{pmatrix}=0$$ quindi sono linearmente dipendenti.

Infatti: $\overline{a}_4=\overline{a}_1+\vec{a}_2+\vec{a}_3$

Possiamo costruire una base al più per $\mathbb{R}^3$:

$\overline{a}_1\cdot\overline{a}_2=-1-1=-2$; $\quad \vec{a}_1\cdot\vec{a}_3=2+1=3$; $\quad \vec{a}_2\cdot\vec{a}_3=1$;

$$\vec{b}_2=\vec{a}_2-\frac{a_2\cdot\vec{a}_1}{|\overline{a}_1|^2}\vec{a}_1=(-1,2,0,1)-\frac{-2}{3}(1,0,1,-1)=$$

$$=(-1,2,0,1)+\left(\frac{2}{3},0,\frac{2}{3},-\frac{2}{3}\right)=\boxed{\left(-\frac{1}{3},2,\frac{2}{3},\frac{1}{3}\right)=\vec{b}_2}$$

$$\boxed{(1,0,1,-1)=\vec{b}_1}$$

$$b_3=\overline{a}_3-\frac{\overline{a_3\cdot b_1}}{|b_1|^2}b_1-\frac{\overline{a_3\cdot b_2}}{|b_2|^2}b_2=$$

$$=(0,1,2,-1)-\frac{3}{3}(1,0,1,-1)-\frac{2+\frac{4}{3}-\frac{1}{3}}{\frac{1}{9}+4+\frac{4}{9}+\frac{1}{9}}\left(-\frac{1}{3},2,\frac{2}{3},\frac{1}{3}\right)=$$

$$=(-1,1,1,0)-\frac{3}{\frac{14}{3}}\left(-\frac{1}{3},2,\frac{2}{3},\frac{1}{3}\right)=$$

N.B.
$b_1\cdot b_2=0 \quad b_1\cdot b_3=0$
$b_2\cdot b_3=0$

$$=(-1,1,1,0)+\left(\frac{3}{14},-\frac{9}{7},-\frac{3}{7},-\frac{3}{14}\right)=\boxed{\left(-\frac{11}{14},-\frac{2}{7},\frac{4}{7},-\frac{3}{14}\right)=\vec{b}_3}$$

III

5) $A=\begin{pmatrix}-2&0&0\\0&1&0\\0&0&3\end{pmatrix}$ $\quad \det A = (-2)(1)(3) = -6$

$B=\begin{pmatrix}1&0&2\\-1&1&0\\-2&1&-2\end{pmatrix}$ $\quad \det B = 0$ infatti $r_3 = r_2 - r_1$

$\det AB = \det A \det B = 0$

$A_{11} = (-1)^{1+1}\det\begin{pmatrix}1&0\\0&3\end{pmatrix}$; $A_{12} = (-1)\det\begin{pmatrix}0&0\\0&3\end{pmatrix}$

$$A^{-1} = \frac{cof(A)^T}{\det A}$$

$A_{13}=\det\begin{pmatrix}0&1\\0&0\end{pmatrix}$; $A_{23} = -\det\begin{pmatrix}0&0\\0&3\end{pmatrix}$; $A_{33}=\det\begin{pmatrix}-2&0\\0&1\end{pmatrix}$

$A_{21} = -\det\begin{pmatrix}1&0\\0&3\end{pmatrix}$; $A_{22} = \det\begin{pmatrix}-2&0\\0&3\end{pmatrix}$; $A_{31} = -\det\begin{pmatrix}0&0\\1&0\end{pmatrix}$

$A_{32} = -\det\begin{pmatrix}-2&0\\0&0\end{pmatrix}$;

$$A^{-1} = \frac{1}{\det}\begin{pmatrix}3&0&0\\0&-6&0\\0&0&-2\end{pmatrix}^T = \frac{1}{-6}\begin{pmatrix}3&0&0\\0&-6&0\\0&0&-2\end{pmatrix} = \begin{pmatrix}-1/2&0&0\\0&1&0\\0&0&1/3\end{pmatrix}$$

D'altro canto l'inversa di una matrice diagonale è ancora diagonale i cui elementi sono i reciproci della matrice di partenza.

6) $\begin{pmatrix}0&1&k\\2&k-3&4\\1&k&-k\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1-k\\1+k\\1\end{pmatrix}$ $\quad A\vec{x}=\vec{b}$

$\det A = -1(-2k-4)+k(2k-(k-3)) = 2k+4+k^2+3k =$
$= k^2+5k+4=0 \quad k = \frac{-5\pm\sqrt{25-16}}{2} = \frac{-5\pm3}{2}$

se $k \neq -4, -1$ il sistema ammette soluzione (IV)

$\vec{x} = A^{-1}\vec{b}$

$\boxed{k=-4}$ → non esistono soluzioni.

$\text{rank}\begin{pmatrix} 0 & 1 & 4 \\ 2 & -7 & 4 \\ 1 & -4 & +4 \end{pmatrix} = 2.$ (det $\neq 0$)

$\text{rank}\begin{pmatrix} 0 & 1 & -4 & 5 \\ 2 & -7 & 4 & -3 \\ 1 & -4 & 4 & 1 \end{pmatrix} = 3.$

$\det = 1(4+12)+4(-7-12)+$
$+5(-28+16) = 16 - \cancel{76} - 60 \neq 0$

$\boxed{k=-1}$ → non esistono soluzioni.

$\text{rank}\begin{pmatrix} 0 & 1 & -1 \\ 2 & -4 & 4 \\ 1 & -1 & 1 \end{pmatrix} = 2$ (det $\neq 0$)

$\text{rank}\begin{pmatrix} 0 & 1 & 1 & 2 \\ 2 & -4 & 4 & 0 \\ 1 & -1 & 1 & 1 \end{pmatrix} =$

$\det = 1(4)-1(-4)+2(-4+4) =$
$= 4+4 = 8 \neq 0$

7) $A = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$ $\quad \det\begin{pmatrix} 1-\lambda & 3 & 4 \\ 3 & 1-\lambda & 0 \\ 4 & 0 & 1-\lambda \end{pmatrix} = 0$

$(1-\lambda)\left[(1-\lambda)^2\right] - 3\left[3(1-\lambda)\right] + 4\left[-4(1-\lambda)\right] = 0$

$(1-\lambda)\left[(1-\lambda)^2 - 9 \cancel{+} 16\right] = 0$

V

$(1-\lambda)\left[(1-\lambda)^2-25\right]=0$

$\boxed{\lambda_1=1}$

$1-\lambda=\pm 5 \longrightarrow \boxed{\lambda_2=6} \quad \boxed{\lambda_3=-4}$

$\boxed{\lambda_1=1}$ $\begin{cases} \cancel{x}+3y+4z=\cancel{x} \\ 3x+\cancel{y}=\cancel{y} \\ 4x+\cancel{z}=\cancel{z} \end{cases}$ $\begin{cases} z=-3/4\, y \\ x=0 \\ x=0 \end{cases}$

$\vec{v}_1=\begin{pmatrix} 0 \\ 1 \\ -3/4 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix}$

$\boxed{\lambda_2=6}$ $\begin{cases} x+3y+4z=6x \\ 3x+y=6y \\ 4x+z=6z \end{cases} \longrightarrow \begin{cases} \frac{5}{4}z+3\,\frac{3}{\cancel{5}}\,\frac{\cancel{5}}{4}z+4z=\frac{15}{2}z \\ y=\frac{3}{5}x \\ x=\frac{5}{4}z \end{cases}$

$\left(\frac{5}{4}+\frac{9}{4}+4-\frac{15}{2}\right)z=0 \qquad \frac{5+9+16-30}{4}z=0 \qquad 0=0$

(ovvio!!)

$\vec{v}_2=\begin{pmatrix} 5/4 \\ 3/4 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$

$\boxed{\lambda_3=-4}$ $\begin{cases} x+3y+4z=-4x \\ 3x+y=-4y \\ 4x+z=-4z \end{cases} \longrightarrow \begin{cases} x+3\left(-\frac{3}{5}x\right)+4\left(-\frac{4}{5}x\right)=-4x \\ y=-\frac{3}{5}x \\ z=-\frac{4}{5}x \end{cases}$

$x\left(1-\frac{9}{5}-\frac{16}{5}+4\right)=0 \qquad x\,\frac{5-9-16+20}{5}=0 \qquad 0=0$

(ovvio!!)

$$\boxed{\vec{VI}}$$

$$\vec{v}_3 = \begin{pmatrix} 1 \\ -3/5 \\ -4/5 \end{pmatrix} \longrightarrow \begin{pmatrix} 5 \\ -3 \\ -4 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & 5 & 5 \\ 4 & 3 & -3 \\ -3 & 4 & -4 \end{pmatrix} \qquad D = S^{-1}AS$$

$SD = AS$

$$\begin{pmatrix} 0 & 5 & 5 \\ 4 & 3 & -3 \\ -3 & 4 & -4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -4 \end{pmatrix} \overset{?}{=} \begin{pmatrix} 1 & 3 & 4 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 5 & 5 \\ 4 & 3 & -3 \\ -3 & 4 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 30 & -20 \\ 4 & 18 & 12 \\ -3 & 24 & 16 \end{pmatrix} = \begin{pmatrix} 0 & 5+9+16 & 5-9-16 \\ 4 & 15+3 & 15-3 \\ -3 & 20+4 & 20-4 \end{pmatrix} \quad \text{ok}$$

8) $4x^2+y^2+4xy+2x+y=0$

$$\det\begin{pmatrix} 4 & 2 & 1 \\ 2 & 1 & 1/2 \\ 1 & 1/2 & 1 \end{pmatrix} = 0.$$

Infatti: $(2x+y)^2+2x+y = (2x+y)(2x+y+1)=0$

la conica è degenere

$y=-2x \qquad y=-2x-1$