SVOLGIMENTO PROVA DI GEOMETRIA E ALGEBRA del 19/3/2019

$\boxed{1}$

(1) $y = -x^2 - 4x + 1 \qquad A = (3,5)$

$y - 5 = m(x-3) \qquad y = m(x-3) + 5$

$$\begin{cases} y = -x^2 - 4x + 1 \\ y = m(x-3) + 5 \end{cases} \qquad m(x-3) + 5 = -x^2 - 4x + 1$$

$x^2 + (4+m)x - 3m + 4 = 0$

$\Delta = (4+m)^2 - 4(4-3m) = 0$

$\cancel{16} + 8m + m^2 \cancel{- 16} + 12m = 0$

$m^2 + 20m = 0 \qquad m_1 = 0 \quad e \quad m_2 = -20$

le rette sono $\begin{cases} y = 5 \\ y = -20x + 65 \end{cases}$

$x_V = -\dfrac{b}{2a} = -\dfrac{-4}{2(-1)} = -2$

$V \equiv (-2, 5)$

$y_V = -(-2)^2 - 4(-2) + 1 = -4 + 8 + 1 = 5.$

Siccome $m = 2ax_0 + b \Rightarrow x_0 = \dfrac{m - b}{2a}$

$x_1 = \dfrac{m_1 - b}{2a} = \dfrac{-(-4)}{2(-1)} = -2 \rightarrow y_1 = 5. \quad \boxed{P_1 \equiv V}$

$x_2 = \dfrac{m_2 - b}{2a} = \dfrac{-20-(-4)}{2(-1)} = 8; \quad y_2 = -64 - 32 + 1 = -95 \qquad P_2 \equiv (8, -95)$

Abbiamo solo due punti a disposizione quindi esiste una famiglia di circonferenze.

$$x^2+y^2+ax+by+c=0$$

$$\begin{cases} 4+25-2a+5b+c=0 \\ 64+95^2+8a-95b+c=0 \end{cases} \Rightarrow \begin{cases} a=\dfrac{2(482+c)}{3} \\ b=\dfrac{1841+c}{15} \end{cases}$$

(3) $\vec{a}=(1,1,2)$ e $\vec{b}=(-1,1,1)$

$$\vec{a}\times\vec{b}=\det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix}=\hat{i}(1-2)+\hat{j}(-2-1)+\hat{k}(1+1)=(-1,-3,2)$$

$|\vec{a}\times\vec{b}|=\sqrt{1+9+4}=\sqrt{14}$ (area del parallelogra

$$\begin{cases} \vec{a}_{\parallel}=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\,\vec{b}=\dfrac{(-1+1+2)}{3}(-1,1,1)=\left(-\dfrac{2}{3},\dfrac{2}{3},\dfrac{2}{3}\right) \\ \vec{a}_{\perp}=\vec{a}-\vec{a}_{\parallel}=(1,1,2)-\left(-\dfrac{2}{3},\dfrac{2}{3},\dfrac{2}{3}\right)=\left(\dfrac{5}{3},\dfrac{1}{3},\dfrac{4}{3}\right) \end{cases}$$

$$\vec{a}_{\parallel}\cdot\vec{a}_{\perp}=\left(-\dfrac{2}{3},\dfrac{2}{3},\dfrac{2}{3}\right)\cdot\left(\dfrac{5}{3},\dfrac{1}{3},\dfrac{4}{3}\right)=\dfrac{1}{9}(-10+2+8)=0$$

c.v.d.

③ $\sin(3\alpha) = \sin(2\alpha+\alpha) = \sin 2\alpha \cos\alpha + \cancel{\sin\alpha\cos}\ \sin\alpha \cos 2\alpha$

$= 2\sin\alpha\cos\alpha\cos\alpha + \sin\alpha(\cos^2\alpha - \sin^2\alpha) =$

$= 2\sin\alpha\cos^2\alpha + \sin\alpha\cos^2\alpha - \sin^3\alpha =$

$= 3\sin\alpha\cos^2\alpha - \sin^3\alpha = \sin\alpha(3\cos^2\alpha - \sin^2\alpha) =$

$= \sin\alpha(3\,\cancel{\cos}\,(1-\sin^2\alpha) - \sin^2\alpha)$

$= \sin\alpha(3 - 4\sin^2\alpha).$

⑤ $A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & k & 1 \\ 1 & -1 & 0 \end{pmatrix}$ $\det A = 1(1) + 1(-1) - 1(-k) =$

$= \cancel{1} - \cancel{1} + k = k.$

Se $\det A \neq 0$ $\text{rank} A = 3 \Rightarrow k \neq 0$

se $\det A = 0$ essa per $k = 0$ otteniamo:

$A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \Rightarrow \text{rank} A = 2.$

$B^{-1} = \begin{pmatrix} -3/2 & 1/2 & -1 \\ -1/2 & 1/2 & -1 \\ 1 & 0 & 1 \end{pmatrix}$; $B^{-1}A = \begin{pmatrix} -5/2 & \frac{5+k}{2} & 2 \\ -3/2 & \frac{3+k}{2} & 1 \\ 2 & -2 & -1 \end{pmatrix}$

$\det(B^{-1}A) = -\frac{k}{2} = 0$ $k = 0$ quindi $\text{rank}(B^{-1}A) = 2.$

$\boxed{4}$

6) $\begin{cases} x+y+kz=2 \\ x+y+3z=k-1 \\ 2x+ky-z=1 \end{cases}$ $\begin{pmatrix} 1 & 1 & k \\ 1 & 1 & 3 \\ 2 & k & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ k-1 \\ 1 \end{pmatrix}$

$\det\begin{pmatrix} 1 & 1 & k \\ 1 & 1 & 3 \\ 2 & k & -1 \end{pmatrix} = 6-5k+k^2=0 \quad k = \begin{cases} 2 \\ 3 \end{cases}$

Se $k \neq 2,3$ $\exists !$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 & 1 & k \\ 1 & 1 & 3 \\ 2 & k & -1 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ k-1 \\ 1 \end{pmatrix}=$

$=\begin{pmatrix} \frac{k^2+2k}{k-2} \\ \frac{2(k+2)}{k-2} \\ -1 \end{pmatrix}$;

Se $k=2$ rank $\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 3 & 1 \\ 2 & 2 & -1 & 1 \end{pmatrix}=3.$

incompatibile.

Se $k=3$ rank $\begin{pmatrix} 1 & 1 & 3 & 2 \\ 1 & 1 & 3 & 2 \\ 2 & 3 & -1 & 1 \end{pmatrix}=2.$

compatibile. $\exists$ ∞^1 soluzioni.

$\begin{cases} x+y+3z=2 \\ x+y+3z=2 \\ 2x+3y-z=1 \end{cases} \Rightarrow \begin{cases} x+y+3z=2 \\ 2x+3y-z=1 \end{cases}$

$$\begin{cases} x = 2 - y - 3z \\ 4 - 2y - 6z + 3y - z = 1 \end{cases} \quad \begin{cases} x = 2 - y - 3z \\ y - 7z = -3 \end{cases}$$

$$\begin{cases} x = 2 - (7z - 3) - 3z \\ y = 7z - 3 \end{cases} \quad \begin{cases} x = 5 - 10z \\ y = 7z - 3 \end{cases} \quad \begin{cases} x = 5 - 10z \\ y = 7z - 3 \\ z \text{ arbitrario} \end{cases}$$

(7) $A = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 5 & -1 \\ 3 & 0 & 2 \end{pmatrix}$; $\det \begin{pmatrix} 2-\lambda & 0 & 3 \\ 1 & 5-\lambda & -1 \\ 3 & 0 & 2-\lambda \end{pmatrix} =$

$$= (2-\lambda)\left\{ (5-\lambda)(2-\lambda) \right\} + 3\left\{ -(5-\lambda)3 \right\} = 0$$

$$(2-\lambda)^2(5-\lambda) - 9(5-\lambda) = 0$$

$$(5-\lambda)\left((2-\lambda)^2 - 9\right) = 0$$

$\nearrow$ $5 - \lambda = 0 \quad \lambda = 5$.

$\searrow$ $2 - \lambda = \pm 3 \quad \lambda = 2 \mp 3 = \begin{cases} 5 \\ -1 \end{cases}$

$\lambda_1 = -1 \quad m = 1$.

$\lambda_2 = 5 \quad m = 2$

$\lambda_1 = -1 \implies \begin{cases} 2x + 3z = -x \\ x + 5y - z = -y \\ 3x + 2z = -z \end{cases} \quad \begin{cases} x = -z \\ 6y = 2z \\ \hline \end{cases}$

$$\begin{cases} x = -z \\ y = z/3 \end{cases} \qquad \vec{v_1} = (-1, 1/3, 1).$$

$$\lambda_2 = 5 \Rightarrow \begin{cases} 2x + 3z = 5x \\ x + 5y - z = 5y \\ 3x + 2z = 5z \end{cases} \qquad \begin{cases} x = z \\ \cancel{\quad} \\ \cancel{\quad} \end{cases}$$

$$\vec{v} = (x, y, x) = x(1,0,1) + y(0,1,0)$$

$$\vec{v_2} = (1,0,1) \quad e \quad \vec{v_3} = (0,1,0).$$

$$\det\begin{pmatrix} -1 & 1 & 0 \\ 1/3 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \neq 0$$

$\vec{v_1}, \vec{v_2}, \vec{v_3}$ sono linearmente indipendenti.

La matrice A è diagonalizzabile.

(8) $x^2 + 4y^2 - 4xy - 4x + 3y + 1 = 0$

$$A = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 3/2 \\ -2 & 3/2 & 1 \end{pmatrix} \qquad A' = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

$$\det A = 1\left(4 - \frac{9}{4}\right) + 2(-2+3) - 2(-3+8) =$$

$$= \frac{7}{4} + 2 - 10 \neq 0$$

$\boxed{17}$

$\det A' = 4 - 4 = 0$ Si tratta di una parabola.

$(1-\lambda)(4-\lambda) - 4 = 0$ $\cancel{4} - \lambda - 4\lambda + \lambda^2 \cancel{-4} = 0$

$\lambda^2 - 5\lambda = 0$ $\lambda = 0$ e $\lambda = 5$.

$\begin{cases} x - 2y = 0 \\ -2x + 4y = 0 \end{cases} \Rightarrow x = 2y$ $\vec{v_1} = (2, 1)$

$\begin{cases} x - 2y = 5x \\ -2x + 4y = 5y \end{cases} \Rightarrow$ $y = -2x$ $\vec{v_2} = (1, -2)$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \Rightarrow \begin{cases} x = 2x' + y' \\ y = x' - 2y' \end{cases}$

$(2x'+y')^2 + 4(x'-2y')^2 - 4(2x'+y')(x'-2y') - 4(2x'+y') +$

$+ 3(x'-2y') + 1 = 0$

$\cancel{4x'^2} + \cancel{4x'y'} + y'^2 \cancel{+4x'^2} \cancel{-16x'y'} + 16y'^2 \cancel{-8x'^2} \cancel{+16x'y'} \cancel{-4x'y'} + 8y'^2 +$

$-8x' - 4y' + 3x' - 6y' + 1 = 0$

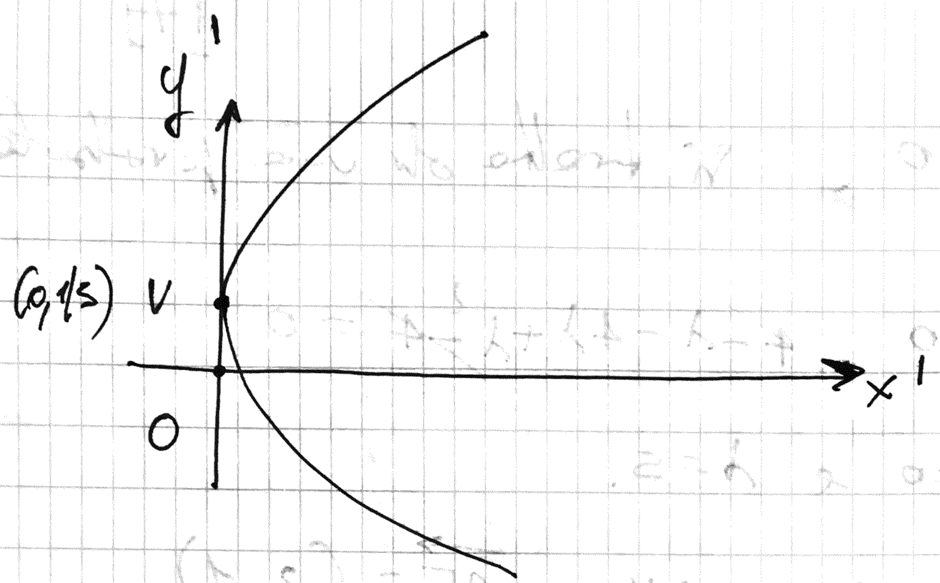
$-5x' + 25y'^2 - 10y' + 1 = 0$

$x' = 5y'^2 - 2y' + 1/5$

$y_v = -\frac{-2}{10} = 1/5$

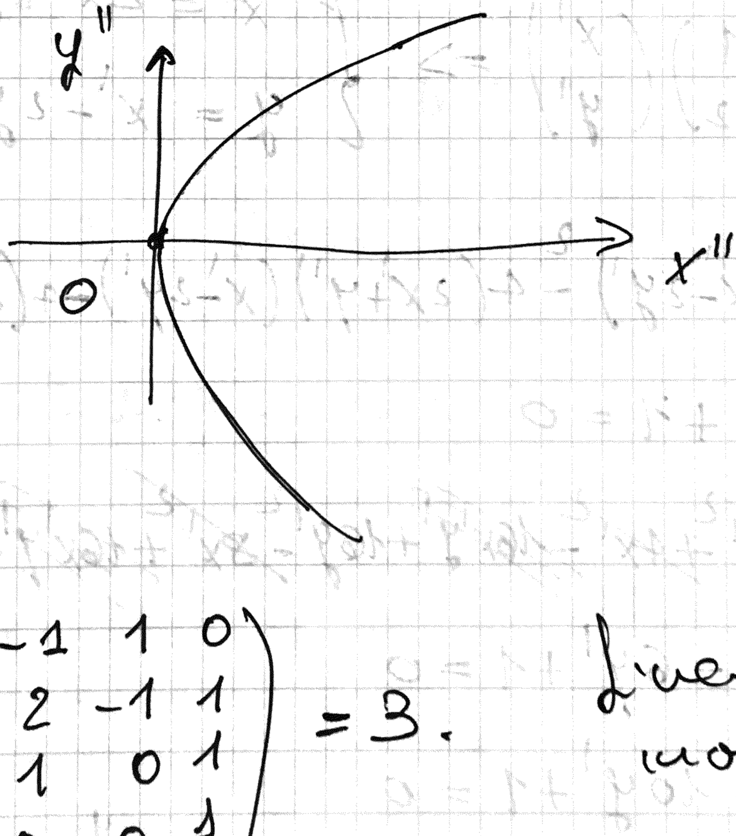
$x_v = \frac{1}{5} - \frac{2}{5} + \frac{1}{5} = 0$

$V \equiv (0, 1/5)$



$\cancel{x' = \frac{1}{5}\left(\frac{x^2}{25}\right)}$ $\qquad x' = \frac{1}{5}\left(25y'^2 - 10y' + 1\right) = \frac{1}{5}\left(5y' - 1\right)^2$

$$\begin{cases} x' = x'' \\ 5y' - 1 = y'' \end{cases} \Rightarrow \boxed{x'' = \frac{1}{5} y''^2}$$



(4)

$$\det\begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 2 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} = 3.$$

Linearmente indipendenti.

$$\vec{b}_1 \doteq \vec{a}_1 = (1,-1,1,0). \qquad |\vec{b}_1| = \sqrt{3}$$

$$\vec{b}_2 = \vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{b}_1}{|\vec{b}_1|^2}\vec{b}_1 = (1,2,-1,1) - \frac{-2}{3}(1,-1,1,0)$$

$$= (1,2,-1,1) + \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, 0\right) = \left(\frac{5}{3}, \frac{4}{3}, -\frac{1}{3}, 1\right)$$

$$\vec{b}_1 \cdot \vec{b}_2 = (1,-1,1,0)\cdot\left(\frac{5}{3}, \frac{4}{3}, -\frac{1}{3}, 1\right) = \frac{5}{3} - \frac{4}{3} - \frac{1}{3} = 0.$$

$$\vec{b}_1 \perp \vec{b}_2. \qquad |\vec{b}_2| = \sqrt{51}/3$$

$$\vec{b}_3 = \vec{a}_3 - \frac{\vec{a}_3 \cdot \vec{b}_1}{|\vec{b}_1|^2}\vec{b}_1 - \frac{\vec{a}_3 \cdot \vec{b}_2}{|\vec{b}_2|^2}\vec{b}_2 =$$

$$= (-1,1,0,1) - \frac{-2}{3}(1,-1,1,0) - \frac{2/3}{51/9}\left(\frac{5}{3}, \frac{4}{3}, -\frac{1}{3}, 1\right)$$

$$= (-1,1,0,1) + \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, 0\right) + \left(-\frac{10}{51}, -\frac{8}{51}, \frac{2}{51}, -\frac{6}{51}\right)$$

$$=$$