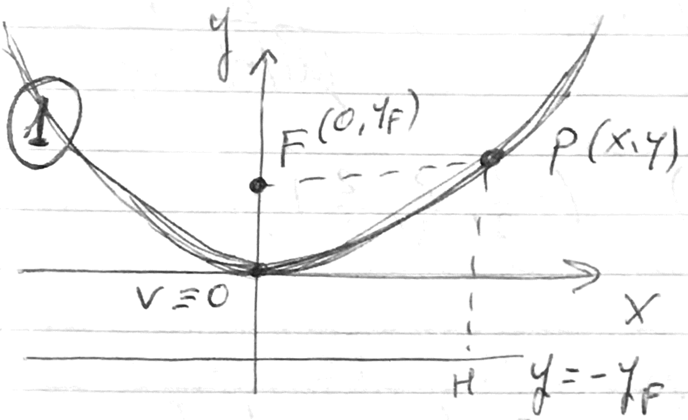
PROVA SCRITTA DI ALGEBRA E GEOMETRIA DEL 9/4/2019

SVOLGIMENTO

(1)



$\overline{PF} = \overline{PH}$

$$x^2 + (y - y_F)^2 = (y + y_F)^2$$

$$x^2 + y^2 - 2y_F y + y_F^2 = y^2 + 2y_F y + y_F^2$$

$$y = \frac{1}{4y_F}x^2 \Rightarrow \frac{1}{4y_F} \doteq a \Rightarrow \boxed{y_F = \frac{1}{4a} \quad e \quad y = ax^2}$$

Nel caso generale $y = ax^2 + bx + c$ il vertice della parabola ha ascissa $x_V = -b/2a$ da cui l'ordinata è:

$$y_V = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = \frac{b^2}{4a} - \frac{b^2}{2a} + c = -\frac{b^2}{4a} + c = \frac{4ac - b^2}{4a} =$$

$$= -\frac{\Delta}{4a}; \quad V \equiv \left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right);$$

Nel caso generale abbiamo una traslazione dell'origine O al punto V. Quindi

$$\begin{cases} y_F = y_F^{(0)} + y_V = \frac{1}{4a} + \left(-\frac{\Delta}{4a}\right) = \frac{1 - \Delta}{4a} = \frac{1 - b^2 + 4ac}{4a}; \\ x_F = x_F^{(0)} + x_V = 0 + \left(-\frac{b}{2a}\right) = -b/2a = -\frac{b}{2a} \end{cases}$$

c.v.d.

(2)

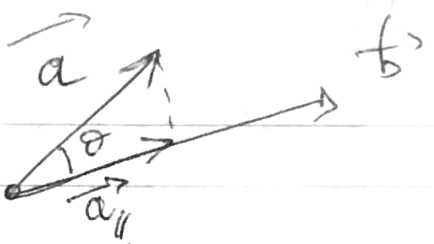
$$tg(2\alpha) = \frac{sen(2\alpha)}{cos(2\alpha)} = \frac{2 sen\alpha cos\alpha}{cos^2\alpha - sen^2\alpha} = \frac{2\, sen\alpha/cos\alpha}{1 - (sen\alpha/cos\alpha)^2} =$$

$$= \frac{2\, tg\alpha}{1 - tg^2\alpha}, \quad \text{con } \alpha \neq \pi/2 \text{ e } \pi/4;$$

(3)

$$\vec{a} \times (\vec{a} - \vec{b}) = -\vec{a} \times \vec{b} = \vec{b} \times \vec{a} = (-1, 1, 1) \times (-1, 1, 2) =$$

$$= (1, 3, -2); \quad A = |\vec{a} \times \vec{b}| = |\vec{b} \times \vec{a}| = \sqrt{1 + 9 + 4} = \sqrt{14}$$



$$\vec{a}_{\parallel} = |\vec{a}|\cos\theta \, \hat{b} = |\vec{a}| \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \frac{\vec{b}}{|\vec{b}|} =$$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = \frac{-1+1+2}{3}(-1,1,1) = \left(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right);$$

(4)

$$\det\begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix} = 1(-1-2) = -3 \neq 0. \quad \text{L.I.}$$

$$\vec{b}_1 \doteq (1,0,0)$$

$$\vec{b}_2 = -\frac{(1,-1,1)\cdot(1,0,0)}{|(1,0,0)|^2}(1,0,0) + (1,-1,1) = (1,-1,1) - (1,0,0) =$$

$$= (0,-1,1)$$

$$\vec{b}_3 = (1,2,1) - \frac{(1,2,1)\cdot(1,0,0)}{|(1,0,0)|^2}(1,0,0) - \frac{(1,2,1)\cdot(0,-1,1)}{|(0,-1,1)|^2}(0,-1,1)$$

$$= (1,2,1) - (1,0,0) - \frac{-2+1}{2}(0,-1,1)$$

$$= (1,2,1) - (1,0,0) + \left(0, -\frac{1}{2}, \frac{1}{2}\right) = (0, 3/2, 3/2)$$

$$\vec{b}_1\cdot\vec{b}_2 = \vec{b}_1\cdot\vec{b}_3 = \vec{b}_2\cdot\vec{b}_3 = 0.$$

$$\boxed{\hat{b}_1 = (1,0,0); \quad \hat{b}_2 = \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right); \quad \hat{b}_3 = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right);}$$

⑤ $A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & k-1 & 1 \\ 1 & -1 & k \end{pmatrix} \Rightarrow \det A = k^2 - 1.$

se $k \neq \pm 1$, Rank $A = 3$.

se $k = \pm 1$, Rank $A = 2$.

$\nexists$ k per cui Rank $A = 1$.

$\det(B^{-1}A) = \det(B^{-1}) \det A = \frac{\det A}{\det B} = \frac{k^2-1}{\det B}$;

$\det B = -2$

se $k \neq \pm 1$ $\det(B^{-1}A) = 0$; $\frac{k^2-1}{\det B} = 2$ $k^2 = (2\det B + 1)$

impossibile $\nexists k \in \mathbb{R}$

⑥ $\begin{pmatrix} -k & k-1 & 1 \\ 0 & k-1 & k \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

$\det \begin{pmatrix} -k & k-1 & 1 \\ 0 & k-1 & k \\ 2 & 0 & 1 \end{pmatrix} = (k-1)(-k-2) - k(0 - 2(k-1)) =$

$= -(k-1)(k+2) + 2k(k-1) = (k-1)(2k-k-2) = (k-1)(k-2) = 0$

$k = 1, 2$.

$k \neq 1, 2$. $\exists!$ soluzione $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -k & k-1 & 1 \\ 0 & k-1 & k \\ 2 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} =$

$= \ldots$; $x = \frac{5(k-1)}{k-2}$, $y = \frac{5k^2+k-2}{(k-1)(k-2)}$, $z = \frac{5k}{2-k}$, se $k=1$ $\begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 5 \end{pmatrix}$ → rank = 2 $\begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 5 \end{pmatrix}$ → rank = 3.

∄ soluzioni.

se $k=2$ $\begin{pmatrix} -2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 2 & 0 & 1 & 5 \end{pmatrix}$ → rank = 2 $\begin{pmatrix} -2 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 5 \end{pmatrix}$ → rank = 3.

∄ soluzione.

(1) $\det\begin{pmatrix} 4-\lambda & 6 & 0 \\ -3 & -5-\lambda & 0 \\ -3 & -6 & -5-\lambda \end{pmatrix} = 0$ $\quad (4-\lambda)\left[+(5+\lambda)^2-0\right]-6\left[3(5+\lambda)-0\right]=$

$= (4-\lambda)(5+\lambda)^2-18(5+\lambda) = (5+\lambda)\left[(4-\lambda)(5+\lambda)-18\right] =$

$(5+\lambda)(20+4\lambda-5\lambda-\lambda^2-18) = (5+\lambda)(2-\lambda-\lambda^2) = 0$

$\lambda=-5$, $\quad \lambda^2+\lambda-2=0 \quad \lambda = \frac{-1\pm\sqrt{1+8}}{2} = \frac{-1\pm 3}{2} = \begin{cases} -2 \\ 1 \end{cases}$

$\boxed{\lambda_1 = 1;\ \lambda_2 = -2;\ \lambda_3 = -5}$

$\begin{cases} 4x+6y = \lambda x \\ -3x-5y = \lambda y \\ -3x-6y-5z = \lambda z \end{cases}$ $\quad \lambda_1 = 1$ $\begin{cases} 3x+6y = 0 \\ -3x+6y-6z = 0 \end{cases}$ $\quad \begin{cases} x = -2y \\ 6y-6y-6z=0 \end{cases}$

$z=0$ e $x=-2y$.

$$\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda_2 = -2$ $\quad \begin{cases} x + y = 0 \\ 3x + 6y + 3z = 0 \end{cases} \quad \begin{cases} x = -y \\ -3y + 6y + 3z = 0 \end{cases} \quad \begin{cases} x = -y \\ z = -y \end{cases}$

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$\lambda_3 = -5$ $\quad \begin{cases} 3x + 2y = 0 \\ x = 0 \\ x + 2y = 0 \end{cases} \quad \begin{cases} y = 0 \\ x = 0 \end{cases}$

$$\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$S = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ e chiaramente S^{-1} esiste. $\vec{v}_1, \vec{v}_2, \vec{v}_3$ sono linearmente indipendenti

(8) $x^2 + 4xy + (k-1)y^2 - 25 = 0$

$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & k-1 & 0 \\ 0 & 0 & -25 \end{pmatrix} \Rightarrow -25\left[(k-1) - 4\right] = -25(k-5) \neq 0$

quindi con $k \neq 5$ la conica non degenere

$A' = \begin{pmatrix} 1 & 2 \\ 2 & k-1 \end{pmatrix} \Rightarrow (k-1) - 4 = k - 5;$

$k > 5$ Ellisse

$k < 5$ Iperbole

$k = 5$ è esclusa.

La conica è già centrata, quindi serve solo una rotazione. Calcoliamo gli autovettori di A'

$$A' = \begin{pmatrix} 1 & 2 \\ 2 & k-1 \end{pmatrix} \qquad \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & k-1-\lambda \end{pmatrix} = 0$$

$$\begin{cases} x + 2y = \lambda x \\ 2y + (k-1)y = \lambda y \end{cases} \Rightarrow \quad y = \frac{(1-\lambda)}{2}x \,; \quad \vec{v}_1 = \begin{pmatrix} 1 \\ \frac{1-\lambda_1}{2} \end{pmatrix} e \ \vec{v}_2 = \begin{pmatrix} 1 \\ \frac{1-\lambda_2}{2} \end{pmatrix}$$

dove λ_1 e λ_2 sono gli autovalori.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1-\lambda_1}{2} & \frac{1-\lambda_2}{2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \Rightarrow \begin{cases} x = x' + y' \\ y = \frac{1-\lambda_1}{2}x' + \frac{1-\lambda_2}{2}y' \end{cases}$$

Sostituiamo nella conica:

$$(x'+y')^2 + 4(x'+y')\left(\frac{\lambda_1-1}{2}x' + \frac{\lambda_2-1}{2}y'\right) + (k-1)\left(\frac{\lambda_1-1}{2}x' + \frac{\lambda_2-1}{2}y'\right)^2 - 25 = 0$$

$$x'^2\left(1 + 2(\lambda_1-1) + (k-1)\frac{(\lambda_1-1)^2}{4}\right) + y'^2\left(1 + 2(\lambda_2-1) + \frac{(k-1)(\lambda_2-1)^2}{4}\right) +$$

$$+ x'y'\left(2 + 2(\lambda_2-1) + 2(\lambda_1-1) + \frac{(k-1)(\lambda_1-1)(\lambda_2-1)}{2}\right) - 25 = 0$$

λ_1 e λ_2 sono tali che $(1-\lambda)(k-1-\lambda) - 4 = 0$

$$k - 1 - \cancel{\lambda} - k\lambda + \cancel{\lambda} + \lambda^2 - 4 = 0$$

$$\lambda^2 - k\lambda + k - 5 = 0 \qquad \lambda = \frac{k \pm \sqrt{k^2 - 4(k-5)}}{2} =$$

$$= \frac{u \pm \sqrt{u^2 - 4u + 20}}{2}$$

$u^2 - 4u + 20 \geq 0$ $\quad u = 2 \pm \sqrt{4 - 20} = 2 \pm \sqrt{-16} = 2 \pm 4i$;

A. V.

d_1 e d_2 sono sempre Reali !!!

$$d_1 = \frac{u + \sqrt{u^2 - 4u + 20}}{2} \quad e \quad d_2 = \frac{u - \sqrt{u^2 - 4u + 20}}{2}$$

Chiaramente il termine $x'y'$ deve "scomparire". Quindi

$$\left[\frac{u-1}{4} d_1^2 - \frac{u-5}{2} d_1 + \frac{u-5}{4}\right] x'^2 + \left[\frac{u-1}{4} d_2^2 - \frac{u-5}{2} d_2 + \frac{u-5}{4}\right] y'^2 = 25$$

$$\frac{1}{100}\left[(u-1) d_1^2 - 2(u-5) d_1 + (u-5)\right] x'^2 + \frac{1}{100}\left[(u-1) d_2^2 - 2(u-5) d_2 + (u-5)\right] y'^2 = 1$$

OK