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The PPN formalism and a general perturbation scheme applied to *f(R)*–theory♪

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Abstract♪

Higher order theories of gravity have recently attracted a lot of ♪ interest as alternative candidates to explain the observed cosmic ♪ acceleration, the flatness of the rotation curves of spiral galaxies ♪ and other relevant astrophysical phenomena. We present the PPN ♪ formalism for applying it to *f(R)*–theory with spherically symmetric ♪ metric. The analogy between *f(R)*–theory and tensor–scalar ♪ theory is shown. With this approach it's possible to give some ♪ contraint to Lagrangian. At last with a perturbative technique ♪ we find some solutions for a class of Lagrangians. Fundamental is ♪ the difference between post–Newtonian and post–Minkowskian ♪ limit.♪

Summary♪

- Newtonian mechanics, General Relativity (GR) with Newtonian and post-Newtonian limit of GR♪
- Parametrized post-Newtonian (PPN) formalism♪
- Difference and equivalence between weak field limit (or Minkowskian and post-Minkowskian limit) and Newtonian limit♪
- *f(R)* – theory of gravity: general concepts♪
- *f(R)* – theory as tensor-scalar theory♪
- A general perturbation scheme for *f(R)*-theory♪
- Newtonian limit for *f(R)*-theory with spherically symmetries for metric♪
- Conclusions♪

Newtonian Mechanics

It's possible to summarize the classical mechanics of gravity as following. The motion equation of particle with mass is

$$m_i \ddot{\mathbf{x}} = -\frac{GMm_g}{|\mathbf{x}|^2}\hat{\mathbf{x}} = -m_g \nabla\varphi(\mathbf{x}) \qquad \text{and if} \qquad m_g = m_i$$

one has $\ddot{\mathbf{x}} = -\nabla\varphi(\mathbf{x})$ where $\varphi(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}$ is the gravitational potential generated by pointlike mass. If the space dependence of potential at "infinity" is as one of pointlike mass $\varphi(\mathbf{x}) \propto |\mathbf{x}|^{-1}$ it's always verified the Gauss's theorem:

$$\Delta\varphi(\mathbf{x}) = 4\pi G\rho(\mathbf{x}) \qquad \text{("local version")} \qquad \int_\Sigma \nabla\varphi(\mathbf{x})\cdot d\mathbf{\Sigma} \propto M \qquad \text{("global version")}$$

At last, the field equations for gravitational field and geodesic motion can be obtained by variational principle

$$\delta\int dt\left[\frac{1}{2}|\nabla\varphi(\mathbf{x})|^2 + 4\pi G\rho(\mathbf{x})\varphi(\mathbf{x})\right] = 0 \qquad \delta\int dt\left[\frac{1}{2}|\dot{\mathbf{x}}|^2 - \varphi(\mathbf{x})\right] = 0$$

The theory of General Relativity♪

The GR starts from equivalence principle point of view. The gravitational field is ♪ identified by the metric tensor ♪$g_{\mu\nu}(x^\lambda)$. The dynamical evolution of metric tensor is♪ governed by Einstein's equation♪

$$\begin{cases} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{\chi}{2}T_{\mu\nu} \\ R = -\frac{\chi}{2}T \end{cases} \qquad \chi = \frac{16\pi G}{c^4}$$

While, the geodesic motion equation is ♪

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\sigma\tau}\frac{dx^\sigma}{ds}\frac{dx^\tau}{ds} = 0$$

where. $ds^2 = g_{\sigma\tau}dx^\sigma dx^\tau$ is the relativistic invariant. The actions for two field equations♪ are♪

$$\delta\int d^4x\sqrt{-g}(R + \chi\mathcal{L}_m) = 0$$

$$\delta\int ds = \delta\int\left(g_{\sigma\tau}dx^\sigma dx^\tau\right)^{1/2} = \delta\int ds\left[g_{00}\left(\frac{dx^0}{ds}\right)^2 + 2g_{0m}\frac{dx^0}{ds}\frac{dx^m}{ds} + g_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\right]^{1/2} = 0$$

The newtonian limit of GR – I♪

The previous geodesic equation can be recast as following♪

$$\frac{d^2x^i}{dx^{0\,2}} = -\Gamma^i_{00} - 2\Gamma^i_{0m}\frac{dx^m}{dx^0} - \Gamma^i_{mn}\frac{dx^m}{dx^0}\frac{dx^n}{dx^0} + \left[\Gamma^0_{00} + 2\Gamma^0_{0m}\frac{dx^m}{dx^0} + 2\Gamma^0_{mn}\frac{dx^m}{dx^0}\frac{dx^n}{dx^0}\right]\frac{dx^i}{dx^0}$$

In the Newtonian context one treats all velocities as vanishingly small in a weak stationary gravitational field and keeps only terms of first order in the difference (obviously) between metric tensor and one of Minkowskian case.♪

$$\frac{d^2x^i}{dx^{0\,2}} \simeq -\Gamma^i_{00} = \frac{1}{2}g_{00}{}^{,i} = -\frac{1}{2}g_{00,i} \qquad\qquad \ddot{\mathbf{x}} = -\frac{c^2}{2}\nabla g_{00}(\mathbf{x})$$

and by remembering the field equation in Newtonian mechanics one has ♪

$$\nabla\left[\frac{c^2}{2}g_{00}(\mathbf{x}) - \varphi(\mathbf{x})\right] = 0 \qquad\qquad g_{00}(\mathbf{x}) = 1 + \frac{2\varphi(\mathbf{x})}{c^2} = 1 - \frac{2GM}{c^2|\mathbf{x}|} = 1 - \frac{r_S}{|\mathbf{x}|}$$

At last the complete metric tensor is♪

$$g_{\mu\nu} = \begin{pmatrix} 1 + \frac{2\varphi(\mathbf{x})}{c^2} & \vec{0}^T \\ \vec{0} & -\gamma_{ij} \end{pmatrix}$$

where♪ $||\gamma_{ij}|| > 0$

The newtonian limit of GR – II

Consider a system of particles that, like the Sun and planets, are bound together by their mutual gravitational attraction. Let $< M >$, $< |\mathbf{x}| >$ and $< |\dot{\mathbf{x}}| >$ be typical values of the masses, separations, and velocities of these particles. It is a familiar result of Newtonian mechanics that the typical kinetic energy will be roughly of the same order of magnitude as the typical potential energy, so

$$< |\dot{\mathbf{x}}| >^2 \sim \frac{G < M >}{< |\mathbf{x}| >}$$

Then, one can declare that Newtonian limit of GR is an approximation of theory at second order (with respect to velocity square). The velocity is given in units of light speed! The Newtonian gravitational potential is of order "two" and the 00 – component of metric tensor is of the same order. The Lagrangian and equation of motion at order two are the same of Newtonian mechanics:

$$\delta \int dx^0 \left(1 + \frac{2\varphi(\mathbf{x})}{c^2} - \left|\frac{d\mathbf{x}}{dx^0}\right|^2\right)^{1/2} = 0 \qquad ds \sim dx^0$$

In the solar system, the Newtonian gravitational potential is nowhere larger than 10^{-5} (in geometrized units).

The post-Newtonian limit of GR I

The post-Newtonian approximation may be described as a method for obtaining the motion of the system to one higher power of the small parameters (gravitational potential or velocity square) than the one given by Newtonian mechanics. So by starting from geodesic equation we need to work out the acceleration $\frac{d^2\mathbf{x}}{d{x^0}^2} \sim \ddot{\mathbf{x}}$ at fourth order. Or alternatively the Lagrangian of geodesic have be known at fourth order. In fact, generally, one has

$$\delta \int ds \simeq \delta \int dx^0 \left(g_{00} + 2g_{0m}\frac{dx^m}{dx^0} + g_{mn}\frac{dx^m}{dx^0}\frac{dx^m}{dx^0} \right)^{1/2} = 0$$

and in our case

$$\delta \int dx^0 \left(1 + g^{(2)}_{00} + g^{(4)}_{00} + 2g^{(3)}_{0m}\frac{dx^m}{dx^0} - \left|\frac{d\mathbf{x}}{dx^0}\right|^2 + g^{(2)}_{mn}\frac{dx^m}{dx^0}\frac{dx^m}{dx^0} \right)^{1/2} = 0$$

and for metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 + g^{(2)}_{00} + g^{(4)}_{00} & g^{(3)}_{0j} \\ g^{(3)}_{i0} & -\gamma_{ij} + g^{(2)}_{ij} \end{pmatrix}$$

The post-Newtonian limit of GR II♪

In computing the Christoffel symbols we must take into account the fact that the♪ temporal distance is bigger than space distance:. $dx^0 \gg |d\mathbf{x}|$ with the consequence♪ that♪

$$\frac{|\partial_0|}{|\nabla|} \sim \mathrm{O}(1) \qquad \partial_0 \sim \dot{\mathbf{x}} \cdot \nabla$$

The same concept must be applied also to Ricci tensor components, Ricci scalar and so on. The metric in the post-Newtonian limit is♪

$$\begin{cases} g_{00}(x_0, \mathbf{x}) \simeq 1 + g^{(2)}_{00}(x_0, \mathbf{x}) + g^{(4)}_{00}(x_0, \mathbf{x}) + \mathrm{O}(6) \\ g_{0i}(x_0, \mathbf{x}) \simeq g^{(3)}_{0i}(x_0, \mathbf{x}) + \mathrm{O}(5) \\ g_{ij}(x_0, \mathbf{x}) \simeq -\delta_{ij} + g^{(2)}_{ij}(x_0, \mathbf{x}) + \mathrm{O}(4) \end{cases}$$

$$\begin{cases} g^{00}(x_0, \mathbf{x}) \simeq 1 + g^{(2)00}(x_0, \mathbf{x}) + g^{(4)00}(x_0, \mathbf{x}) + \mathrm{O}(6) \\ g^{0i}(x_0, \mathbf{x}) \simeq g^{(3)0i}(x_0, \mathbf{x}) + \mathrm{O}(5) \\ g^{ij}(x_0, \mathbf{x}) \simeq -\delta_{ij} + g^{(2)ij}(x_0, \mathbf{x}) + \mathrm{O}(4) \end{cases}$$

$$g^{\mu\sigma} g_{\sigma\nu} = \delta^\mu_\nu$$

$$\begin{cases} g^{(2)00}(x_0, \mathbf{x}) = -g^{(2)}_{00}(x_0, \mathbf{x}) \\ g^{(4)00}(x_0, \mathbf{x}) = g^{(2)}_{00}(x_0, \mathbf{x})^2 - g^{(4)}_{00}(x_0, \mathbf{x}) \\ g^{(3)0i} = g^{(3)}_{0i} \\ g^{(2)ij}(x_0, \mathbf{x}) = -g^{(2)}_{ij}(x_0, \mathbf{x}) \end{cases}$$

The post-Newtonian limit of GR III♪

The Christoffel symbols and Ricci tensor components are shown♪

$$\begin{cases} \Gamma^{(3)}{}^{0}_{00} = \frac{1}{2} g^{(2)}_{00,0} & \Gamma^{(2)}{}^{i}_{00} = \frac{1}{2} g^{(2)}_{00,i} \\ \Gamma^{(2)}{}^{i}_{jk} = \frac{1}{2}\left(g^{(2)}_{jk,i} - g^{(2)}_{ij,k} - g^{(2)}_{ik,j} \right) & \Gamma^{(3)}{}^{0}_{ij} = \frac{1}{2}\left(g^{(3)}_{0i,j} + g^{(3)}_{0j,i} - g^{(2)}_{ij,0} \right) \\ \Gamma^{(3)}{}^{i}_{0j} = \frac{1}{2}\left(g^{(3)}_{0j,i} - g^{(3)}_{i0,j} - g^{(2)}_{ji,0} \right) & \Gamma^{(4)}{}^{0}_{0i} = \frac{1}{2}\left(g^{(4)}_{00,i} - g^{(2)}_{00} g^{(2)}_{00,i} \right) \\ \Gamma^{(4)}{}^{i}_{00} = \frac{1}{2}\left(g^{(4)}_{00,i} + g^{(2)}_{in} g^{(2)}_{00,n} - 2 g^{(3)}_{i0,0} \right) & \Gamma^{(2)}{}^{0}_{0i} = \frac{1}{2} g^{(2)}_{00,i} \end{cases}$$

$$\begin{cases} R^{(2)}_{00} = \frac{1}{2} \triangle_\eta g^{(2)}_{00} \\ R^{(4)}_{00} = \frac{1}{2} \triangle_\eta g^{(4)}_{00} + \frac{1}{2} g^{(2)}_{mn,m} g^{(2)}_{00,n} + \frac{1}{2} g^{(2)}_{mn} g^{(2)}_{00,mn} + \frac{1}{2} g^{(2)}_{mm,00} - \frac{1}{4} g^{(2)}_{00,m} g^{(2)}_{00,m} - \frac{1}{4} g^{(2)}_{mm,n} g^{(2)}_{00,n} - g^{(3)}_{m0,m0} \\ R^{(3)}_{0i} = \frac{1}{2} \triangle_\eta g^{(3)}_{0i} - g^{(2)}_{mi,m0} - g^{(3)}_{m0,mi} + g^{(2)}_{mm,0i} \\ R^{(2)}_{ij} = \frac{1}{2} \triangle_\eta g^{(2)}_{ij} - \frac{1}{2} g^{(2)}_{im,mj} - \frac{1}{2} g^{(2)}_{00,ij} - \frac{1}{2} g^{(2)}_{jm,mi} + \frac{1}{2} g^{(2)}_{mm,ij} \end{cases}$$

The post newtonian limit of GR IV

If one chooses the harmonic gauge $g^{\sigma\tau}\Gamma^{\mu}_{\sigma\tau} = 0$ One obtains simplified expressions for the Ricci tensor and scalar:

$$\begin{cases} R^{(2)}_{00} = \frac{1}{2}\triangle_\eta g^{(2)}_{00} \\ R^{(4)}_{00} = \frac{1}{2}\triangle_\eta g^{(4)}_{00} + \frac{1}{2}g^{(2)}_{mn}g^{(2)}_{00,mn} - \frac{1}{2}g^{(2)}_{00,00} - \frac{1}{2}|\nabla_\eta\, g^{(2)}_{00}|^2 \\ R^{(3)}_{0i} = \frac{1}{2}\triangle_\eta g^{(3)}_{0i} \\ R^{(2)}_{ij} = \frac{1}{2}\triangle_\eta g^{(2)}_{ij} \end{cases}$$

$$\begin{cases} R^{(2)} = R^{(2)}_{00} - R^{(2)}_{mm} = \frac{1}{2}\triangle_\eta(g^{(2)}_{00} - g^{(2)}_{mm}) \\ R^{(4)} = R^{(4)}_{00} - g^{(2)}_{00}R^{(2)}_{00} - g^{(2)}_{mn}R^{(2)}_{mn} = \frac{1}{2}\triangle_\eta g^{(4)}_{00} - \frac{1}{2}g^{(2)}_{00,00} + \frac{1}{2}g^{(2)}_{mn}\left[g^{(2)}_{00,mn} - \triangle_\eta g^{(2)}_{mn}\right] - \frac{1}{2}|\nabla_\eta\, g^{(2)}_{00}|^2 - \frac{1}{2}g^{(2)}_{00}\triangle_\eta g^{(2)}_{00} \end{cases}$$

A such develop has be performed also for the matter

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$$

From their interpretation as the energy density, momentum density and momentun flux, we expect the following expansions

$$T_{00} \simeq T^{(0)}_{00} + T^{(2)}_{00} + \ldots \quad T_{ij} \simeq T^{(0)}_{ij} + \ldots \quad T_{0i} \simeq T^{(1)}_{0i} + \ldots$$

First solution of GR

The first solution of GR has been the Schwarzschild's solution (1916). The Schwarzschild metric is the solution of Einstein's equations with spherical symmetry. In standard coordinates one has

$$ds^2 = \left(1 - \frac{r_S}{r}\right) dx^{0\,2} - \frac{dr^2}{1 - \frac{r_S}{r}} - r^2 d\Omega \qquad \text{in the vacuum}$$

$$ds^2 = \left(1 - \frac{r_S(r)}{r}\right) dx^{0\,2} - \frac{dr^2}{1 - \frac{r_S(r)}{r}} - r^2 d\Omega \qquad \text{in the matter}$$

where $r_S(r) = \dfrac{8\pi G}{c^2} \displaystyle\int_0^r \rho(r') r'^2 dr'$ In isotropic coordinates the metric can be rewritten,

in the vacuum as following

$$ds^2 = \left(\frac{1 - \frac{r_S}{4|\mathbf{x}|}}{1 + \frac{r_S}{4|\mathbf{x}|}}\right)^2 dx^{0\,2} - \left(1 + \frac{r_S}{4|\mathbf{x}|}\right)^4 d\mathbf{x}^2 \qquad r \doteq |\mathbf{x}| \left(1 + \frac{r_S}{4|\mathbf{x}|}\right)^2$$

and by considering that the gravitational potential is of second order $\dfrac{r_S}{|\mathbf{x}|} \ll 1$ one can develop the metric at second order

Approximated Schwarzschild solution / PPN parameters I

$$ds^2 = \left(1 - \alpha\frac{r_S}{|\mathbf{x}|} + \beta\frac{{r_S}^2}{2|\mathbf{x}|^2} + \dots\right)dx^{0^2} - \left(1 + \gamma\frac{r_S}{|\mathbf{x}|} + \dots\right)d\mathbf{x}^2$$

Where the coefficients are the first form of parametrization of a generic solution of GR (obviously with spherical symmetry) in terms of Schwarzschild solution. In fact if these coefficients (Eddington's parameters) are equal to one we find the approximated Schwarzschild solution. Eddington formalism aims at compare the solutions of many theories of gravity amoung them and respect to GR prediction. Generally these parameters depend on the free parameters of the theory (Tensor scalar, Tensor vector, etc). In standard coordinates one has

$$ds^2 = \left(1 - \alpha\frac{r_S}{r} + \frac{\beta - \alpha\gamma}{2}\frac{{r_S}^2}{r^2} + \dots\right)dx^{0^2} - \left(1 + \gamma\frac{r_S}{r} + \dots\right)dr^2 - r^2 d\Omega \qquad r \doteq |\mathbf{x}|\left(1 + \gamma\frac{r_S}{2|\mathbf{x}|} + \dots\right)$$

Then, one can state that the Eddingotn parameters measure how several metric theories can be compared by using Eddingotn's formalism.

Approximated Schwarzschild solution / PPN parameters II♪

In detail we have♪

- γ measures the amount of curvature of space produced by a body of mass♪ M at radius ♪ $|\mathbf{x}|$ in the sense that the space components of the Riemann curvature tensor are♪ given to post−Newtonian order by♪ $R_{ijkl} = \frac{3}{2}\gamma\frac{r_S}{|\mathbf{x}|^3}N_{ijkl}$ independent of the choice of♪ post−Newtonian gauge and where♪ N_{ijkl} represents the tensor geometric♪ properties.♪
- β is said to measure the amount of non linearity that a given theory puts into the "00" component of the metric when one considers the isotropic coordinates.♪
- α This measure is linked to that of mass (experimental): the definition can be englobed into the mass. Only if it were possible to determine the mass by some independent non gravitational measurement would it makes sense to ask whether in fact the its value.♪

Successively the PPN formalism have been developed by Will, Nordtvedt et all. The fundamental idea to construct a parametrization of metric tensor is considering for "00" component only scalar object, for vectorial part only scalar objects multiplied by a velocity and, at last, for space tensorial part only rotations and tensorial product of vectors. Etc .. ♪

The complete PPN Formalism♪

The metric completely expressed in terms of PPN parameters♪

$$
\begin{aligned}
g_{00} =& -1 + 2U - 2\beta U^2 - 2\xi\Phi_W + (2\gamma + 2 + \alpha_3 + \zeta_1 - 2\xi)\Phi_1 \\
&+2(3\gamma - 2\beta + 1 + \zeta_2 + \xi)\Phi_2 + 2(1 + \zeta_3)\Phi_3 + 2(3\gamma + 3\zeta_4 - 2\xi)\Phi_4 \\
&-(\zeta_1 - 2\xi)\mathcal{A} - (\alpha_1 - \alpha_2 - \alpha_3)w^2 U - \alpha_2 w^i w^j U_{ij} + (2\alpha_3 - \alpha_1)w^i V_i \\
&+O(\epsilon^3), \\
g_{0i} =& -\frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)V_i - \frac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi)W_i \\
&-\frac{1}{2}(\alpha_1 - 2\alpha_2)w^i U - \alpha_2 w^j U_{ij} + O(\epsilon^{5/2}), \\
g_{ij} =& (1 + 2\gamma U + O(\epsilon^2))\delta_{ij}.
\end{aligned}
$$

The complete PPN Formalism

Where the metric potentials are defined as ...

$$\Phi_W = \int \frac{\rho' \rho'' (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \cdot \left(\frac{\mathbf{x}'-\mathbf{x}''}{|\mathbf{x}-\mathbf{x}''|} - \frac{\mathbf{x}-\mathbf{x}''}{|\mathbf{x}'-\mathbf{x}''|} \right) d^3x' d^3x'',$$

$$\mathcal{A} = \int \frac{\rho' [\mathbf{v}' \cdot (\mathbf{x}-\mathbf{x}')]^2}{|\mathbf{x}-\mathbf{x}'|^3} d^3x',$$

$$\Phi_1 = \int \frac{\rho' v'^2}{|\mathbf{x}-\mathbf{x}'|} d^3x',$$

$$\Phi_2 = \int \frac{\rho' U'}{|\mathbf{x}-\mathbf{x}'|} d^3x',$$

$$\Phi_3 = \int \frac{\rho' \Pi'}{|\mathbf{x}-\mathbf{x}'|} d^3x',$$

$$\Phi_4 = \int \frac{p'}{|\mathbf{x}-\mathbf{x}'|} d^3x',$$

$$V_i = \int \frac{\rho' v'_i}{|\mathbf{x}-\mathbf{x}'|} d^3x',$$

$$W_i = \int \frac{\rho' [\mathbf{v}' \cdot (\mathbf{x}-\mathbf{x}')](x-x')_i}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'.$$

$$U = \int \frac{\rho'}{|\mathbf{x}-\mathbf{x}'|} d^3x',$$

$$U_{ij} = \int \frac{\rho' (x-x')_i (x-x')_j}{|\mathbf{x}-\mathbf{x}'|^3} d^3x',$$

Post-Newtonian – post-Minkowskian limit of GR

Then, to compute the complete Newtonian and post-Newtonian limit we need two hypothesis: 1) **the weak field** – we perfom a perturbation analysis with respect to the metric tensor; 2) **the slow motion of particles** – then the relativistic invariant is much more than zero (ds^2). With such a picture the Newtonian mechanics is recovered at the lowest order.

The post-Minkowskian limit is recovered from GR theory if we consider only the weak field hypothesis. In this case the time derivative is of the same order of space derivative. The Laplacian, in the PPN formalism, is replaced by the d'Alembertian. An approximation, for examples, for metric tensor can be

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

(weak perturbation onto Minkowski space).

… but the Newtonian limit does not satisfy the observables …♪

The weak field limit of GR with slow motion is the Newtonian mechanics, and the gravitational potential scales as inverse of distance. But in many cases this trend does not fit the astrophysics observables. For example is not clear the rotation curves of galaxies: the Keplerian trend is not observed. Moreover the rotation velocity is constant with respect to radial distance. The solution to the problem can be the dark matter hypothesis: the space surrounding is filled with some “matter”. But if we modify the gravitational potential♪

$$\varphi(\mathbf{x}) = |\mathbf{x}|^{-1} + \tilde{\varphi}(\mathbf{x})$$

from Gauss theorem we have♪

$$\int_\Sigma \nabla\varphi(\mathbf{x}) \cdot d\mathbf{\Sigma} \sim M + \tilde{M}_\Sigma$$

The Gauss theorem is valid only for Newtonian gravitational potential. Then by changing the theory of gravity the “Newton-like” limit of new theory could be different from♪

$$\varphi(\mathbf{x}) \propto |\mathbf{x}|^{-1}$$

$f(R)$ – theory of gravity: general concepts I

A $f(R)$ – theory starts from modification of Lagrangian with a generic function of Ricci scalar. The variational principle states that

$$\delta \int d^4x\sqrt{-g}(f(R)+\mathcal{X}\mathcal{L}_m) = \int d^4x\sqrt{-g}\left[f'R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - \frac{\mathcal{X}}{2}T_{\mu\nu}\right]\delta g^{\mu\nu} + \int d^4x\sqrt{-g}g_{\mu\nu}f'\delta R^{\mu\nu} = 0$$

but the last term is not zero ··· while in GR it's zero!!!

$$\delta \int d^4x\sqrt{-g}(R+\mathcal{X}\mathcal{L}_m) = \int d^4x\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{\mathcal{X}}{2}T_{\mu\nu}\right]\delta g^{\mu\nu} + \int d^4x\sqrt{-g}g_{\mu\nu}\delta R^{\mu\nu} = 0$$

In GR we omitted the last term since it's a total divergence and with some transformation into surface integral we can neglected it at infinity. But in this case it's present ...

$$\int d^4x\sqrt{-g}g_{\mu\nu}f'\delta R^{\mu\nu} \simeq \int d^4x\sqrt{-g}(-f'_{;\mu\nu} + g_{\mu\nu}\Box f')\delta g^{\mu\nu} = 0$$

The field equations are

$$\begin{cases} f'R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - f'_{;\mu\nu} + g_{\mu\nu}\Box f' = \frac{\mathcal{X}}{2}T_{\mu\nu} \\ 3\Box f' + f'R - 2f = \frac{\mathcal{X}}{2}T \end{cases} \qquad f' = \frac{df(R)}{dR}$$

f(R) – theory of gravity: general concepts II♪

From the trace equation we note the first difference with respect to GR. In the vacuum the Ricci scalar is not vanishing, but it can propagate as Klein – Gordon's equation. In GR, only in the matter, it's possible to have Ricci scalar no vanishing; but also in this case the propagating nature is not possible. The *f(R)* theory is with a non-minimal coupling. Besides if one explicitly writes the covariant derivatives, obtains the complete structure of equation and also their complexity♪

$$\begin{cases} \Box f' + \frac{f'R-2f}{3} = \frac{\mathcal{X}}{6}T \\ \Box f' + \frac{dW(f')}{df'} = \frac{\mathcal{X}}{6}T \end{cases}$$

$$\begin{cases} H_{\mu\nu} = f'R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} + \mathcal{H}_{\mu\nu} = \frac{\mathcal{X}}{2}T_{\mu\nu} \\ H = g^{\sigma\tau}H_{\sigma\tau} = f'R - 2f + \mathcal{H} = \frac{\mathcal{X}}{2}T \end{cases}$$

$$\begin{cases} \mathcal{H}_{\mu\nu} = & -f''\left\{R_{,\mu\nu} - \Gamma^{\sigma}_{\mu\nu}R_{,\sigma} - g_{\mu\nu}\left[\left({g^{\sigma\tau}}_{,\sigma} + g^{\sigma\tau}\ln\sqrt{-g}_{,\sigma}\right)R_{,\tau} + g^{\sigma\tau}R_{,\sigma\tau}\right]\right\} + \\ & -f'''\left(R_{,\mu}R_{,\nu} - g_{\mu\nu}g^{\sigma\tau}R_{,\sigma}R_{,\tau}\right) \\ \mathcal{H} = & g^{\sigma\tau}\mathcal{H}_{\sigma\tau} = 3f''\left[\left({g^{\sigma\tau}}_{,\sigma} + g^{\sigma\tau}\ln\sqrt{-g}_{,\sigma}\right)R_{,\tau} + g^{\sigma\tau}R_{,\sigma\tau}\right] + 3f'''g^{\sigma\tau}R_{,\sigma}R_{,\tau} \end{cases}$$

f(R) – theory as tensor–scalar theory I♪

The *f(R)* theory can be recast as tensor–scalar theory with non–minimal coupling.♪
A general tensor–scalar theory, in the so called Jordan frame, is♪

$$\delta\int d^4x\sqrt{-g}\Big[F(\phi)R+\omega(\phi)\phi^{;\sigma}\phi_{;\sigma}+V(\phi)+\mathcal{X}\mathcal{L}_m\Big]=0$$

The field equations are♪

$$\begin{cases} F(\phi)G_{\mu\nu}-\frac{1}{2}V(\phi)g_{\mu\nu}+\omega(\phi)\Big[\phi_{;\mu}\phi_{;\nu}-\frac{1}{2}\phi^{;\sigma}\phi_{;\sigma}g_{\mu\nu}\Big]-F(\phi)_{;\mu\nu}+g_{\mu\nu}\Box F(\phi)=\frac{\mathcal{X}}{2}T_{\mu\nu} \\ 2\omega(\phi)\Box\phi+\omega_{,\phi}(\phi)\phi^{;\sigma}\phi_{;\sigma}-[F(\phi)R+V(\phi)]_{,\phi}=0 \\ 3\Box F(\phi)-F(\phi)R-2V(\phi)-\omega(\phi)\phi^{;\sigma}\phi_{;\sigma}=\frac{\mathcal{X}}{2}T \\ 2\omega(\phi)\Box\phi+3\Box F(\phi)+[\omega_{,\phi}(\phi)-\omega(\phi)]\phi^{;\sigma}\phi_{;\sigma}-[F(\phi)R+V(\phi)]_{,\phi}-F(\phi)R-2V(\phi)=\frac{\mathcal{X}}{2}T \end{cases}$$

$$\begin{cases} f'G_{\mu\nu}-\frac{f-f'R}{2}g_{\mu\nu}-f'_{;\mu\nu}+g_{\mu\nu}\Box f'=\frac{\mathcal{X}}{2}T_{\mu\nu} \\ \phi G_{\mu\nu}-\frac{1}{2}V(\phi)g_{\mu\nu}-\phi_{;\mu\nu}+g_{\mu\nu}\Box\phi=\frac{\mathcal{X}}{2}T_{\mu\nu} \end{cases}$$

If we choose♩ $F(\phi)=\phi$ $\quad\omega(\phi)=0$ the analogy is♪ possible if♪ $f'=\phi$. *f(R)* corresponds to O'Hanlon's Lagrangian.♪

$f(R)$ – theory as tensor–scalar theory II

... but any non–minimal coupling tensor–scalar theory can be recast in a minimal coupling tensor–scalar theory by the so–called conformal transformation. In fact by introducing a new metric

$$\tilde{g}_{\mu\nu} = A(x^\lambda) g_{\mu\nu} \qquad \text{with} \qquad A(x^\lambda) > 0$$

The previous action becomes (Einstein frame)

$$\delta \int d^4x \sqrt{-\tilde{g}} \left[\Lambda \tilde{R} + \Omega(\psi) \tilde{g}_{\sigma\tau} \psi^{;\sigma} \psi^{;\tau} + W(\psi) + \mathcal{X} \tilde{\mathcal{L}}_m \right] = 0$$

where

$$\begin{cases} F(\phi) A^{-1} = \Lambda \\ \Omega(\psi) \left(\frac{d\psi}{d\phi} \right)^2 = \Lambda \left[\frac{\omega(\phi)}{F(\phi)} - \frac{3}{2} \left(\frac{d \ln F(\phi)}{d\phi} \right)^2 \right] \\ W(\psi) = \frac{\Lambda^2}{F(\phi)^2} V(\phi) \\ \tilde{\mathcal{L}}_m = \frac{\Lambda^2}{F(\phi)^2} \mathcal{L}_m (\Lambda F(\phi)^{-1} \tilde{g}_{\mu\nu}) \end{cases}$$

And the new field equations are

$$\begin{cases} \Lambda \tilde{G}_{\mu\nu} - \frac{1}{2} W(\psi) \tilde{g}_{\mu\nu} + \Omega(\psi) \left[\psi_{;\mu} \psi_{;\nu} - \frac{1}{2} \tilde{g}^{\sigma\tau} \psi_{;\sigma} \psi_{;\tau} \tilde{g}_{\mu\nu} \right] = \frac{\mathcal{X}}{2} \tilde{T}^{\psi}_{\mu\nu} \\ 2\Omega(\psi) \tilde{\Box} \psi + \Omega_{,\psi}(\psi) \tilde{g}^{\sigma\tau} \psi_{;\sigma} \psi_{;\tau} - W_{,\psi}(\psi) = \mathcal{X} \tilde{\mathcal{L}}_{m,\psi} \\ \tilde{R} = -\frac{1}{\Lambda} \left(\frac{\mathcal{X}}{2} \tilde{T}^{\psi} + 2W(\psi) + \Omega(\psi) \tilde{g}^{\sigma\tau} \psi_{;\sigma} \psi_{;\tau} \right) \end{cases}$$

$$\delta \int d^4x \sqrt{-g} [f(R) + \mathcal{X} \mathcal{L}_m] = 0 \rightleftarrows \delta \int d^4x \sqrt{-g} [\phi R + V(\phi) + \mathcal{X} \mathcal{L}_m] = 0$$

f(R) – theory as tensor–scalar theory III

With this approach the decoupling between scalar field and Ricci scalar reduces the complexity of the equation, but in the presence of matter we have a non–minimal coupling of new scalar field with ordinary matter.

By using first the conformal transformation it's possible to link the Eddington's parameters [T. Damour, G. Esposito Farese – Class Quantum Grav. **9** 2093 – 2176 (1992)] with the function coupling

$$\gamma - 1 = -2\frac{\alpha(\psi)^2}{1+\alpha(\psi)^2} \qquad \beta - 1 = \frac{\alpha(\psi)^2}{2[1+\alpha(\psi)^2]^2}\frac{\partial\alpha(\psi)}{\partial\phi}\frac{\partial\phi}{\partial\psi} \qquad \alpha(\psi) = -\frac{F_{,\phi}(\phi)}{F(\phi)}\frac{d\phi}{d\psi}$$

and by using the analogy *f(R)* – tensor scalar [S.Capozziello, A. Troisi – Phys. Rev D **72** 044022 (2005); AS, S. Capozziello, A. Troisi – Modern Physics Letters A **21** 2291– 2301 (2006)] one has

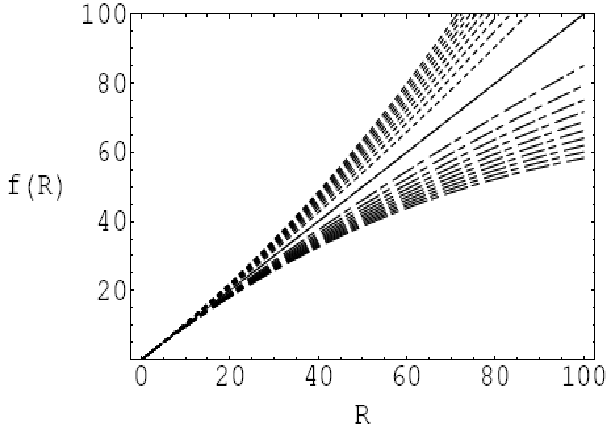
$$\gamma - 1 = -\frac{f''^2}{f' + 2f''^2} \qquad 4(\beta - 1) = \frac{f'f''}{2f' + 3f''^2}\frac{d\gamma}{dR}$$

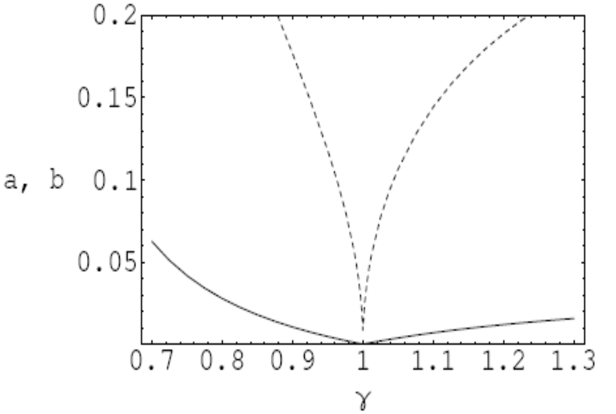
and at last we are able to find a constraint onto analytic expression for *f* with coefficient fixed by Eddington's parameters

$$f_{\pm}(R) \doteq \frac{1}{12}\left|\frac{1-\gamma}{2\gamma-1}\right|R^3 \pm \frac{1}{2}\sqrt{\left|\frac{1-\gamma}{2\gamma-1}\right|}R^2 + R + \Lambda \qquad \beta = 1$$

$f(R)$ – theory as tensor–scalar theory IV

The "solutions" for f–theory are compatible with the Taylor expansion of $f(R)$: the experimental value of Eddington's parameters determines the weight of the square and cubic term.





Plot of the modulus of coefficients: *a* (R^3) (line) and *b* (R^2) (dashed line).

The experimental values of Eddington's parameters

Mercury Perihelion Shift	$\|2\gamma_0^{PPN} - \beta_0^{PPN} - 1\| < 3 \times 10^{-3}$
Lunar Laser Ranging	$4\beta_0^{PPN} - \gamma_0^{PPN} - 3 = -(0.7 \pm 1) \times 10^{-3}$
Very Long Baseline Interferometry	$\|\gamma_0^{PPN} - 1\| = 4 \times 10^{-4}$
Cassini spacecraft	$\gamma_0^{PPN} - 1 = (2.1 \pm 2.3) \times 10^{-5}$

Spherically symmetric solution with constant scalar curvature for *f(R)*-theory I

Let us consider a metric with spherical time-dependent symmetry

$$ds^2 = A(t,r)dt^2 - B(t,r)dr^2 - r^2 d\Omega$$

and hypothesize a solution with constant Ricci scalar. The field equation becomes

$$\begin{cases} f_0' R_{\mu\nu} - \frac{1}{2} f_0 g_{\mu\nu} = \frac{\mathcal{X}}{2} T_{\mu\nu} \\ f_0' R_0 - 2 f_0 = \frac{\mathcal{X}}{2} T \end{cases} \qquad \begin{matrix} f(R_0) = f_0 \\ f'(R_0) = f_0' \end{matrix} \qquad \begin{cases} R_{\mu\nu} + \lambda g_{\mu\nu} = q \frac{\mathcal{X}}{2} T_{\mu\nu} \\ R_0 = \eta \frac{\mathcal{X}}{2} T - 4\lambda \end{cases} \qquad \begin{matrix} \lambda = -\frac{1}{2} \frac{f_0}{f_0'} \\ q^{-1} = f_0' \end{matrix}$$

for the matter we choose the usual tensor $T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu}$

The solution, if $p = -\rho$, is $ds^2 = \left(1 + \frac{k_1}{r} + \frac{q\mathcal{X}\rho - 2\lambda}{6} r^2\right) dt^2 - \frac{dr^2}{1 + \frac{k_1}{r} + \frac{q\mathcal{X}\rho - 2\lambda}{6} r^2} - r^2 d\Omega$

The only matter, compatible with constant Ricci scalar, is the cosmological constant!
In this case the Birkhoff theorem is verified.

[AS, S. Capozziello, A. Troisi – in preparation]

Spherically symmetric solution with constant scalar curvature for *f(R)*–theory II♪

Since we are analyzing the weak field limit of higher order gravity theories it can be a reasonable approximation to consider that our Lagrangian works as well♪

as the Hilbert–Einstein gravity when♪$R \to 0$ (this even means that we can suitably♪

put the cosmological constant to zero):♪$\lim\limits_{R\to 0} f \sim R$. In such a case the trace equation♪

$f_0' R_0 - 2f_0 = \dfrac{\mathcal{X}}{2} T$ indicates that in the vacuum case one can obtain a class of solutions♪

with constant curvature. In particular they will be solutions with vanishing Ricci scalar which match the prescriptions of GR. Let us now suppose that our Lagrangian,♪ in the limit of small values of the four–dimensional curvature, tends to a constant:♪

$\lim_{R\to 0} f = \Lambda$ Even in this case by looking only at the trace equation one can hold some♪ interesting considerations: It is evident that non–zero curvature solutions are obtainable. In addition, in such a case, even in the absence of matter, there are no Ricci flat spacetimes as solution of the field equations since the higher order derivative terms induce to constant curvature solutions without the cosmological constant. Of course, this is not the case when one considers the GR–theory.♪

Spherically symmetric solution with constant scalar curvature for *f(R)*-theory III

In fact in the standard theory of gravity the constant curvature solutions are in order only in presence of matter because of the proportionality of the Ricci scalar and the trace of the energy-momentum tensor of matter, or on the other side one can get a similar framework in presence of a cosmological constant. Actually the big difference between General Relativity and higher order theories of gravity is that in the general framework the Schwarzschild-de Sitter solution is not necessarily driven by a cosmological term while the effect of an "effective" cosmological constant in the low energy limit can be played by the higher order derivative contributes evaluated on Ricci constant backgrounds.

Let us consider a Lagrangian satisfying the condition $\lim_{R\to 0} f \sim R^2$. This theory does not reproduce GR. In such a case analyzing the whole set of field equations one can observe that for vanishing scalar curvature the equations are solved if the Ricci scalar is vanishing. The class of solution for the metric $ds^2 = a(r)dt^2 - b(r)dr^2 - r^2 d\Omega$ is

$$b(r) = \frac{\exp[-\int dr\, h(r)]}{K + 4\int \frac{dr\, a(r)\exp[-\int dr\, h(r)]}{r[a(r)+ra'(r)]}}$$

where $h(r) = \dfrac{r^2 a'(r)^2 - 4a(r)^2 - 2ra(r)[2a(r)' + ra(r)'']}{ra(r)[4a(r) + ra'(r)]}$

Spherically symmetric solution with costant scalar curvature for *f(R)*–theory IV♪

We report in the following table several fourth order Lagrangians which admit solutions with constant or zero scalar curvature. Each Lagrangian contemplates Schwarzschild, Schwarzschild–de Sitter and the class of solutions.♪

$f(R)$ - theory		Field equations
R	$\longrightarrow$	$R_{\mu\nu}=0$, with $R=0$
$\xi_1 R+\xi_2 R^n$	$\longrightarrow$	$\begin{cases} R_{\mu\nu}=0 & \text{with } R=0,\ \xi_1\neq 0 \\ R_{\mu\nu}+\lambda g_{\mu\nu}=0 & \text{with } R=\left[\frac{\xi_1}{(n-2)\xi_2}\right]^{\frac{1}{n-1}},\ \xi_1\neq 0,\ n\neq 2 \\ 0=0 & \text{with } R=0,\ \xi_1=0 \\ R_{\mu\nu}+\lambda g_{\mu\nu}=0 & \text{with } R=R_0,\ \xi_1=0,\ n=2 \end{cases}$
$\xi_1 R+\xi_2 R^{-m}$	$\longrightarrow$	$R_{\mu\nu}+\lambda g_{\mu\nu}=0$ with $R=\left[-\frac{(m+2)\xi_2}{\xi_1}\right]^{\frac{1}{m+1}}$
$\xi_1 R+\xi_2 R^n+\xi_3 R^{-m}$	$\longrightarrow$	$R_{\mu\nu}+\lambda g_{\mu\nu}=0$, with $R=R_0$ so that $\xi_1 R_0^{m+1}+(2-n)\xi_2 R_0^{n+m}+(m+2)\xi_3=0$
$\frac{R}{\xi_1+R}$	$\longrightarrow$	$\begin{cases} R_{\mu\nu}=0 & \text{with } R=0 \\ R_{\mu\nu}+\lambda g_{\mu\nu}=0 & \text{with } R=-\frac{\xi_1}{2} \end{cases}$
$\frac{1}{\xi_1+R}$	$\longrightarrow$	$R_{\mu\nu}+\lambda g_{\mu\nu}=0$, with $R=-\frac{2\xi_1}{3}$

Spherically symmetric solution with space-dependent scalar curvature for perturbative *f(R)*-theory

If we consider the metric $ds^2 = A(t,r)dt^2 - B(t,r)dr^2 - r^2 d\Omega$ and the *f(R)*-theory as $f(R) = R + \epsilon\Phi(R)$ with $\epsilon \ll 1$ and we try to find solutions with a such constraint $\bar{R} = R(r)$ we have

$$A(t,r) = a(r)\hat{a}(t) \qquad a(r) = \frac{b(r)e^{-\frac{2}{3}\int\frac{[R+\epsilon(2\Phi - R\Phi')]b(r)}{\epsilon R'\Phi''}dr}}{\epsilon^2 r^4 R'^2 \Phi''^2}$$

$$B(t,r) = b(r) \qquad b(r) = -\frac{3(rR'\Phi'')_{,r}}{rR}\epsilon$$

In this case the Birkhoff's theorem is verified. The crucial component to demonstrate it is the same of GR: The only off-diagonal equation non identically vanishing is

$$\frac{d}{dr}\left(r^2 f'\right)\dot{B}(t,r) = 0$$

The other choice is not possible: we would find the incompatibility.

A perturbative approach to *f(R)* theory I

Let us suppose for the metric an expression $g_{\mu\nu}=g^{(0)}_{\mu\nu}+g^{(1)}_{\mu\nu}$ and compute the perturbative development into field equations … and in *f(R)*−theory:

$$f=\sum_n \frac{f^{n(0)}}{n!}\left[R-R^{(0)}\right]^n \approx \sum_n \frac{f^{n(0)}}{n!}R^{(1)^n} \approx f^{(0)}+f'^{(0)}R^{(1)}$$

$$f = f^{(0)}+f'^{(0)}R^{(1)}\,,\quad f'=f'^{(0)}+f''^{(0)}R^{(1)}\,,\quad f''=f''^{(0)}+f'''^{(0)}R^{(1)}\,,\qquad f'''=f'''^{(0)}+f^{IV\,(0)}R^{(1)}$$

at zero order we have $f'^{(0)}R^{(0)}_{\mu\nu}-\frac{1}{2}g^{(0)}_{\mu\nu}f^{(0)}+\mathcal{H}^{(0)}_{\mu\nu} = \frac{\mathcal{X}}{2}T^{(0)}_{\mu\nu}$

where $\mathcal{H}^{(0)}_{\mu\nu} = -f''^{(0)}\left\{R^{(0)}_{,\mu\nu}-\Gamma^{(0)\rho}{}_{\mu\nu}R^{(0)}_{,\rho}-g^{(0)}_{\mu\nu}\left(g^{(0)\rho\sigma}{}_{,\rho}R^{(0)}_{,\sigma}+g^{(0)\rho\sigma}R^{(0)}_{,\rho\sigma}+g^{(0)\rho\sigma}\ln\sqrt{-g}_{,\rho}R^{(0)}_{,\sigma}\right)\right\}+$

$$-f'''^{(0)}\left\{R^{(0)}_{,\mu}R^{(0)}_{,\nu}-g^{(0)}_{\mu\nu}g^{(0)\rho\sigma}R^{(0)}_{,\rho}R^{(0)}_{,\sigma}\right\}.$$

A perturbative approach to *f(R)* theory II

At first order we have ♪ $f'^{(0)}\left\{R^{(1)}_{\mu\nu}-\frac{1}{2}g^{(0)}_{\mu\nu}R^{(1)}\right\}+f''^{(0)}R^{(1)}R^{(0)}_{\mu\nu}-\frac{1}{2}f^{(0)}g^{(1)}_{\mu\nu}+\mathcal{H}^{(1)}_{\mu\nu}\;=\;\frac{\mathcal{X}}{2}T^{(1)}_{\mu\nu}$

where ♪

$$\begin{aligned}
\mathcal{H}^{(1)}_{\mu\nu}\;=\;&-f''^{(0)}\Big\{R^{(1)}_{,\mu\nu}-\Gamma^{(0)\rho}{}_{\mu\nu}R^{(1)}_{,\rho}-\Gamma^{(1)\rho}{}_{\mu\nu}R^{(0)}_{,\rho}-g^{(0)}_{\mu\nu}\Big[g^{(0)\rho\sigma}{}_{,\rho}R^{(1)}_{,\sigma}+g^{(1)\rho\sigma}{}_{,\rho}R^{(0)}_{,\sigma}+g^{(0)\rho\sigma}R^{(1)}_{,\rho\sigma}+\\
&+g^{(1)\rho\sigma}R^{(0)}_{,\rho\sigma}+g^{(0)\rho\sigma}\Big(\ln\sqrt{-g}^{(0)}_{,\rho}R^{(1)}_{,\sigma}+\ln\sqrt{-g}^{(1)}_{,\rho}R^{(0)}_{,\sigma}\Big)+g^{(1)\rho\sigma}\ln\sqrt{-g}^{(0)}_{,\rho}R^{(0)}_{,\sigma}\Big]-g^{(1)}_{\mu\nu}\Big(g^{(0)\rho\sigma}{}_{,\rho}R^{(0)}_{,\sigma}+\\
&+g^{(0)\rho\sigma}R^{(0)}_{,\rho\sigma}+g^{(0)\rho\sigma}\ln\sqrt{-g}^{(0)}_{,\rho}R^{(0)}_{,\sigma}\Big)\Big\}-f'''^{(0)}\Big\{R^{(0)}_{,\mu}R^{(1)}_{,\nu}+R^{(1)}_{,\mu}R^{(0)}_{,\nu}-g^{(0)}_{\mu\nu}g^{(0)\rho\sigma}\Big(R^{(0)}_{,\rho}R^{(1)}_{,\sigma}+R^{(1)}_{,\rho}R^{(0)}_{,\sigma}\Big)+\\
&-g^{(0)}_{\mu\nu}g^{(1)\rho\sigma}R^{(0)}_{,\rho}R^{(0)}_{,\sigma}-g^{(1)}_{\mu\nu}g^{(0)\rho\sigma}R^{(0)}_{,\rho}R^{(0)}_{,\sigma}\Big\}-f'''^{(0)}R^{(1)}\Big\{R^{(0)}_{,\mu\nu}-\Gamma^{(0)\rho}{}_{\mu\nu}R^{(0)}_{,\rho}-g^{(0)}_{\mu\nu}\Big(g^{(0)\rho\sigma}{}_{,\rho}R^{(0)}_{,\sigma}+\\
&+g^{(0)\rho\sigma}R^{(0)}_{,\rho\sigma}+g^{(0)\rho\sigma}\ln\sqrt{-g}_{,\rho}R^{(0)}_{,\sigma}\Big)\Big\}-f^{IV\,(0)}R^{(1)}\Big\{R^{(0)}_{,\mu}R^{(0)}_{,\nu}-g^{(0)}_{\mu\nu}g^{(0)\rho\sigma}R^{(0)}_{,\rho}R^{(0)}_{,\sigma}\Big\}\,.
\end{aligned}$$

A perturbative approach to *f(R)* theory III

The *f(R)*−theory can be “perturbated” also independently from the metric. we can think it as a little displacement from GR theory. In this case the considered approach can be simplified♪

$$f = R + \Phi \qquad \Phi \ll 1 \qquad f = R^{(0)} + R^{(1)} + \Phi^{(0)}, \quad f' = 1 + \Phi'^{(0)}, \quad f'' = \Phi'''^{(0)}, \quad f''' = \Phi'''^{(0)}$$

and for the metric the same expression ♪ $g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu}$

At zero order we have ♪ $R^{(0)}_{\mu\nu} - \frac{1}{2}R^{(0)}g^{(0)}_{\mu\nu} = \frac{\mathcal{X}}{2}T^{(0)}_{\mu\nu}$

In this approach the solution at zero order is that of GR. In previous case it’s not true!♪

At first order we have ♪ $R^{(1)}_{\mu\nu} - \frac{1}{2}g^{(0)}_{\mu\nu}R^{(1)} - \frac{1}{2}g^{(1)}_{\mu\nu}R^{(0)} - \frac{1}{2}g^{(0)}_{\mu\nu}\Phi^{(0)} + \Phi'^{(0)}R^{(0)}_{\mu\nu} + \mathcal{H}^{(1)}_{\mu\nu} = \frac{\mathcal{X}}{2}T^{(1)}_{\mu\nu}$

where ♪

$$\mathcal{H}^{(1)}_{\mu\nu} = -\Phi'''^{(0)}\left\{R^{(0)}_{,\mu}R^{(0)}_{,\nu} - g^{(0)}_{\mu\nu}g^{(0)rr}R^{(0)}_{,r}R^{(0)}_{,r}\right\} - \Phi''^{(0)}\left\{R^{(0)}_{,\mu\nu} - \Gamma^{(0)r}_{\mu\nu}R^{(0)}_{,r} - g^{(0)}_{\mu\nu}\left({g^{(0)}}^{rr}_{,r}R^{(0)}_{,r} + \right.\right.$$

$$\left.\left. + g^{(0)rr}R^{(0)}_{,rr} + g^{(0)rr}\ln\sqrt{-g}^{(0)}_{,r}R^{(0)}_{,r}\right)\right\}.$$

A perturbative approach to *f(R)* theory IV

f - theory	$\Lambda + R + \epsilon R \ln R$
metric	$a(r) = b(r)^{-1} = 1 + \frac{k_1}{r} - \frac{\Lambda r^2}{6} + \delta x(r)$
solutions	$x(r) = \frac{k_2}{r} + \frac{\epsilon\Lambda[\ln(-2\Lambda)-1]r^2}{6\delta}$
metric	$a(r) = 1 - \frac{\Lambda r^2}{6} + \delta x(r), \quad b(r) = \frac{1}{1-\frac{\Lambda r^2}{6}} + \delta y(r)$
solutions	$\begin{cases} x(r) = (\Lambda r^2 - 6)\left\{k_1 + \int dr \frac{4\delta(2\Lambda^2 r^4 - 15\Lambda r^2 + 18)y(r) + r\{36r\epsilon\Lambda[\log(-2\Lambda)-1] + \delta(\Lambda r^2 - 6)^2 y'(r)\}}{36 r\delta(\Lambda r^2 - 6)}\right\} \\ y(r) = \frac{k_2\delta - 6r^3\epsilon\Lambda[\ln(-2\Lambda)-1]}{r\delta(r^2\Lambda - 6)^2} \end{cases}$
f - theory	$R + \epsilon R^n$
metric	$a(r) = b(r)^{-1} = 1 + \frac{k_1}{r} + \delta x(r)$
solutions	$x(r) = \frac{k_2}{r}$
metric	$a(r) = 1 + \delta\frac{x(r)}{r}, \quad b(r) = 1 + \delta\frac{y(r)}{r}$
solutions	$x(r) = k_1 + k_2 r, \quad y(r) = k_3$

A perturbative approach to *f(R)* theory V

f - theory	R^n
metric	$a(r)=b(r)^{-1}=1+\frac{k_1}{r}+\frac{R_0r^2}{12}+\delta x(r)$
solution	$\begin{cases} n=2, & R_0\neq 0 \text{ and } x(r)=\frac{3k_2-k_3}{3r}+\frac{k_3r^2}{12}+\frac{k_4}{r}\int dr\,r^2\left\{\int dr\frac{\exp\left[\frac{R_0r_0^2\ln(r-r_0)}{8+3R_0r_0^2}\right]}{r^5}\right\} \\ & \text{with } r_0 \text{ satisfying the condition } 6k_1+8r_0+R_0r_0^3=0 \\ n\geq 2, & \text{System solved only whit } R_0=0 \text{ and no prescriptions on } x(r) \end{cases}$
metric	$a(r)=1+\delta\frac{x(r)}{r},\quad b(r)=1+\delta\frac{y(r)}{r}$
solutions	$\begin{cases} n=2 & y(r)=-\frac{R_0r^3}{6}-\frac{x(r)}{2}+\frac{1}{2}rx'(r)+k_1,\ R(r)=\delta R_0 \\ n\neq 2 & y(r)=-\frac{1}{2}\int dr\,r^2R(r)-\frac{x(r)}{2}+\frac{1}{2}rx'(r)+k_1 \text{ with } R(r) \text{ whatever} \end{cases}$
metric	$a(r)=1-\frac{r_g}{r}+\delta x(r),\quad b(r)=\frac{1}{1-\frac{r_g}{r}}+\delta y(r)$
solution	$\begin{cases} n=2 & y(r)=\frac{rk_1}{3r_g^2-7r_gr+4r^2}+\frac{r^2k_2}{3(3r_g^2-7r_gr+4r^2)}+\frac{r_gr^2x(r)+2(r_gr^3-r^4)x'(r)}{(3r_g-4r)(r_g-r)^2} \\ n\neq 2 & \text{whatever functions } x(r)\,,\ y(r) \text{ and } R(r) \end{cases}$

A perturbative approach to *f(R)* theory VI

f - theory	$R/(\xi+R)$
metric	$a(r)=1+\delta\frac{x(r)}{r},\quad b(r)=1+\delta\frac{y(r)}{r}$
solutions	$\begin{cases} x(r)=-\frac{4e^{-\frac{\xi^{1/2}r}{\sqrt{6}}}}{\xi}k_1-\frac{2\sqrt{6}e^{\frac{\xi^{1/2}r}{\sqrt{6}}}}{\xi^{3/2}}k_2+k_3r \\ y(r)=-\frac{2e^{-\frac{\xi^{1/2}r}{\sqrt{6}}}(6\xi^{1/2}+\sqrt{6}\xi r)}{3b^{3/2}}k_1-\frac{2e^{\frac{\xi^{1/2}r}{\sqrt{6}}}(\sqrt{6}-\xi^{1/2}r)}{\xi^{3/2}}k_2 \end{cases}$

The last 11 slides belong to a paper in preparation [AS, S. Capozziello, A. Troisi]♪

Newtonian limit of *f(R)* theory with spherically symmetric metric I

We start from the metric

$$\begin{cases} g_{tt}(t,r) = A(t,r) \simeq 1 + g^{(2)}_{tt}(t,r) + g^{(4)}_{tt}(t,r) \\ g_{rr}(t,r) = -B(t,r) \simeq -1 + g^{(2)}_{rr}(t,r) \\ g_{\theta\theta}(t,r) = -r^2 \\ g_{\phi\phi}(t,r) = -r^2 \sin^2\theta \end{cases}$$

The contravariant forms are

$$\begin{cases} g^{tt} = A(t,r)^{-1} \simeq 1 - g^{(2)}_{tt} + [{g^{(2)}_{tt}}^2 - g^{(4)}_{tt}] \\ g^{rr} = -B(t,r)^{-1} \simeq -1 - g^{(2)}_{rr} \end{cases}$$

The Christoffel symbols are

$$\begin{cases} {\Gamma^{(3)}}^t_{tt} = \frac{g^{(2)}_{tt,t}}{2} & {\Gamma^{(2)}}^r_{tt} + {\Gamma^{(4)}}^r_{tt} = \frac{g^{(2)}_{tt,r}}{2} + \frac{g^{(2)}_{rr}g^{(2)}_{tt,r} + g^{(4)}_{tt,r}}{2} \\ {\Gamma^{(3)}}^r_{tr} = -\frac{g^{(2)}_{rr,t}}{2} & {\Gamma^{(2)}}^t_{tr} + {\Gamma^{(4)}}^t_{tr} = \frac{g^{(2)}_{tt,r}}{2} + \frac{g^{(4)}_{tt,r} - g^{(2)}_{tt}g^{(2)}_{tt,r}}{2} \\ {\Gamma^{(3)}}^t_{rr} = -\frac{g^{(2)}_{rr,t}}{2} & {\Gamma^{(2)}}^r_{rr} + {\Gamma^{(4)}}^r_{rr} = -\frac{g^{(2)}_{rr,r}}{2} - \frac{g^{(2)}_{rr}g^{(2)}_{rr,r}}{2} \\ \Gamma^r_{\phi\phi} = \sin^2\theta\,\Gamma^r_{\theta\theta} & {\Gamma^{(0)}}^r_{\theta\theta} + {\Gamma^{(2)}}^r_{\theta\theta} + {\Gamma^{(4)}}^r_{\theta\theta} = -r - r g^{(2)}_{rr} - r {g^{(2)}_{rr}}^2 \end{cases}$$

The determinant is

$$g \simeq r^4 \sin^2\theta \{-1 + [g^{(2)}_{rr} - g^{(2)}_{tt}] + [g^{(2)}_{tt}g^{(2)}_{rr} - g^{(4)}_{tt}]\}$$

Newtonian limit of *f(R)* theory with spherically symmetric metric II♪

The components of Ricci tensor are♪

$$
\begin{cases}
R_{tt}^{(2)} = \frac{r g_{tt,rr}^{(2)} + 2 g_{tt,r}^{(2)}}{2r} \\[2ex]
R_{tt}^{(4)} = \frac{-r {g_{tt,r}^{(2)}}^{2} + 4 g_{tt,r}^{(4)} + r g_{tt,r}^{(2)} g_{rr,r}^{(2)} + 2 g_{rr}^{(2)} [2 g_{tt,r}^{(2)} + r g_{tt,rr}^{(2)}] + 2 r g_{tt,rr}^{(4)} + 2 r g_{rr,tt}^{(2)}}{4r} \\[2ex]
R_{tr}^{(3)} = -\frac{g_{rr,t}^{(2)}}{r} \\[2ex]
R_{rr}^{(2)} = -\frac{r g_{tt,rr}^{(2)} + 2 g_{rr,r}^{(2)}}{2r} \\[2ex]
R_{\theta\theta}^{(2)} = -\frac{2 g_{rr}^{(2)} + r [g_{tt,r}^{(2)} + g_{rr,r}^{(2)}]}{2}
\end{cases}
$$

The any *f(R)* can be developed as Taylor series in some point♪

$$
f(R) = \sum_n \frac{f^n(R_0)}{n!}(R - R_0)^n \simeq f_0 + f_1 R + f_2 R^2 + f_3 R^3 + \ldots
$$

The Ricci scalar is♪

$$
\begin{cases}
R^{(2)} = \frac{2 g_{rr}^{(2)} + r [2 g_{tt,r}^{(2)} + 2 g_{rr,r}^{(2)} + r g_{tt,rr}^{(2)}]}{r^2} \\[2ex]
R^{(4)} = \frac{4 {g_{rr}^{(2)}}^{2} + 2 r g_{rr}^{(2)} [2 g_{tt,r}^{(2)} + 4 g_{rr,r}^{(2)} + r g_{tt,rr}^{(2)}] + r\{-r {g_{tt,r}^{(2)}}^{2} + 4 g_{tt,r}^{(4)} + r g_{tt,r}^{(2)} g_{rr,r}^{(2)} - 2 g_{tt}^{(2)} [2 g_{tt,r}^{(2)} + r g_{tt,rr}^{(2)}] + 2 r g_{tt,rr}^{(4)} + 2 r g_{rr,tt}^{(2)}\}}{2r^2}
\end{cases}
$$

Newtonian limit of *f(R)* theory with spherically symmetric metric III

By resolving the equation at second order we obtain

$$\begin{cases} g_{tt}^{(2)} = \delta_0 + \tilde{\delta}_1(t)\frac{e^{-\tilde{r}}}{\tilde{r}} + \tilde{\delta}_2(t)\frac{e^{\tilde{r}}}{\tilde{r}} \\ g_{rr}^{(2)} = -\tilde{\delta}_1(t)\frac{(\tilde{r}+1)e^{-\tilde{r}}}{\tilde{r}} + \tilde{\delta}_2(t)\frac{(\tilde{r}-1)e^{\tilde{r}}}{\tilde{r}} \\ R^{(2)} = \frac{3}{l^2}\left[\tilde{\delta}_1(t)\frac{e^{-\tilde{r}}}{\tilde{r}} + \tilde{\delta}_2(t)\frac{e^{\tilde{r}}}{\tilde{r}}\right] \end{cases} \qquad f_0 = 0 \qquad l^2 = \left|\frac{6f_2}{f_1}\right| \qquad \tilde{r} = \frac{r}{l}$$

Where $\tilde{\delta}_1(t)$, $\tilde{\delta}_2(t)$ are some time dependent integration constant. At third order we have $f_1 g_{rr,t}^{(2)} + 2f_2 r R_{,tr}^{(2)} = 0$. This equation states the relation between the "rr" component and Ricci scalar. In fact, generally, if the "rr" component is time-independent and the "00" component is the product of two functions (one of time and other one of the space) the Ricci scalar is time independent. Indeed exists every a redefinition of time gives back a metric time-independent.

Is not verified the Birkhoff theorem in *f(R)*? It would seem ...

Post-Minkowskian limit of *f(R)* theory with spherically symmetric metric I♪

By using the perturbative approach, considered previously, we can consider♪

$$\begin{cases} A(t,r) = 1 + a(t,r) \\ B(t,r) = 1 + b(t,r) \end{cases} \qquad a(t,r), b(t,r) \ll 1$$

The field equations are♪

$$\begin{cases} f_0 = 0 \\ f_1 \left\{ R^{(1)}_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} R^{(1)} \right\} + \mathcal{H}^{(1)}_{\mu\nu} = 0 \end{cases}$$

where♪

$$\mathcal{H}^{(1)}_{\mu\nu} = -f_2 \left\{ R^{(1)}_{,\mu\nu} - \Gamma^{(0)\rho}{}_{\mu\nu} R^{(1)}_{,\rho} - g^{(0)}_{\mu\nu} \left[g^{(0)\rho\sigma}{}_{,\rho} R^{(1)}_{,\sigma} + g^{(0)\rho\sigma} R^{(1)}_{,\rho\sigma} + g^{(0)\rho\sigma} \ln \sqrt{-g}^{(0)}_{,\rho} R^{(1)}_{,\sigma} \right] \right\}$$

For great values of radial coordinate we have♪

$$\begin{cases} \frac{\partial^2 a(t,r)}{\partial r^2} - \frac{\partial^2 b(t,r)}{\partial t^2} = 0 \\ f_1 \Big[a(t,r) - b(t,r) \Big] - 8 f_2 \left[\frac{\partial^2 b(t,r)}{\partial r^2} + \frac{\partial^2 a(t,r)}{\partial t^2} - 2 \frac{\partial^2 b(t,r)}{\partial t^2} \right] = \Psi(t) \end{cases}$$

Post-Minkowskian limit of *f(R)* theory with spherically symmetric metric II

The solution is some "gravitational wave"

$$\begin{cases} a(t,r) = \int \frac{d\omega dk}{2\pi}\tilde{a}(\omega,k)e^{i(\omega t-kr)} \\ b(t,r) = \int \frac{d\omega dk}{2\pi}\tilde{b}(\omega,k)e^{i(\omega t-kr)} \end{cases}$$

$$\begin{cases} \tilde{a}(\omega,k) = \tilde{b}(\omega,k)\,, & \omega = \pm k \\ \tilde{a}(\omega,k) = \left[1-\frac{3\xi}{4k^2}\right]\tilde{b}(\omega,k)\,, & \omega = \pm\sqrt{k^2-\frac{3\xi}{4}} \end{cases}$$

The propagation modes are two: one massless (similar to GR) and an other one **massive.** The similar outcome in the tensorial perturbation of metric have been considered in a paper in preparation [AS, S. Capozziello, A. Troisi]

The last 5 slides belong to paper on preparation [AS, S. Capozziello, A. Troisi]

Conclusions♪

The fundamental topics have been:♪

♪

• The PPN formalism: general concepts. Its application for spherically symmetric metric have been applied♪

♪

• Approach to *f(R)*–theory as generalization of GR–theory: principal aspect and some solution for particular cases have been shown. Exact solution is found if we consider a constant Ricci scalar (Birkhoff theorem is valid) – there is an approximated solution if we consider Ricci scalar space dependent.♪

• A general perturbative scheme have been applied and post–Minkowskian limit have been recovered♪