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The Weak Field Limit of Higher Order Gravity

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The Weak Field Limit of Higher Order Gravity

Higher Order Theories of gravity have recently attracted a lot of interest as alternative candidates to explain the observed cosmic acceleration, the flatness of the rotation curves of spiral galaxies and other relevant astrophysical phenomena. It is a crucial point testing these alternative theories in the weak field (Newtonian, Minkowskian) limit. In general, it is possible to find out spherically symmetric solutions and compare them with those of General Relativity (GR). We find Yukawa-like corrections in the Newtonian case and a massive gravitational waves in the Minkowskian case. As soon as the modifications of theory are removed, we find the usual outcomes of GR. Also the Noether symmetries technique has been investigated to find new time-independent spherically symmetric solutions.

Summary

- *f*–theory of gravity: general concepts;
- Spherical symmetry in *f*-gravity with constant scalar curvature;
- The Noether Symmetry approach to *f*-gravity;
- Spherically symmetric solution with space-dependent scalar curvature for *f*-gravity by perturbations;
- A perturbative approach to *f*-gravity;
- Eddington parameters and *f*-theory by scalar-tensor theory analogy;
- Newtonian limit of *f*-theory in vacuum with spherically symmetric metric (standard coordinates);
- Newtonian limit of quadratic theory of gravity in presence of matter with spherically symmetric metric (isotropic coordinates);
- *f*-gravity and O'Hanlon theory;
- Post – Minkowskian limit of *f*-gravity: gravitational waves in *f*–gravity;
- Energy-momentum tensor of *f*-gravity;
- Conclusions;
- Perspectives.

f–gravity: general concepts

f-gravity takes into account modifications of Hilbert-Einstein Lagrangian assuming a generic function of curvature invariants (in particular the Ricci scalar). The field equations are

$$H_{\mu\nu} \doteq f'R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - f'_{;\mu\nu} + g_{\mu\nu}\Box f' = \mathcal{X}T_{\mu\nu}$$

$$H = g^{\alpha\beta}H_{\alpha\beta} = 3\Box f' + f'R - 2f = \mathcal{X}T$$

We have fourth order differential equations.

From the trace equation in vacuum, it is possible to find out non-vanishing solutions.

As shown by Starobinsky, the Ricci scalar propagates as a Klein-Gordon field.

Spherically symmetric solutions

We are interested to investigate solutions in the Newtonian limit. The most general symmetric solution can be written as follows

$$ds^2 = g_1(t, |\mathbf{x}|)\, dt^2 + g_2(t, |\mathbf{x}|)\, dt\, \mathbf{x} \cdot d\mathbf{x} + g_3(t, |\mathbf{x}|)(\mathbf{x} \cdot d\mathbf{x})^2 + g_4(t, |\mathbf{x}|) d|\mathbf{x}|^2$$

but we have an arbitrariness in the choice of coordinates. Then we can use also the expressions of metric (isotropic coordinates):

$$ds^2 = g_{tt}(t', r')\, dt^2 - g_{ij}(t', r') dx^i dx^j$$

and (standard coordinates):

$$ds^2 = g_{tt}(t', r'')\, dt^2 - g_{rr}(t', r'') dr''^2 - r''^2 d\Omega$$

Spherical symmetry in *f*-gravity

If the metric is time-independent, the definition of Ricci scalar gives rise to a differential equation (Bernoulli equation):

$$b'(r)+\left\{\frac{r^2a'(r)^2-4a(r)^2-2ra(r)[2a(r)'+ra(r)'']}{ra(r)[4a(r)+ra'(r)]}\right\}b(r)$$

$$+\left\{\frac{2a(r)}{r}\left[\frac{2+r^2R(r)}{4a(r)+ra'(r)}\right]\right\}b(r)^2\doteq b'(r)+h(r)b(r)+l(r)b(r)^2=0$$

This is a relation linking the metric potentials and the Ricci scalar.

If we consider theories without cosmological constant + Hilbert-Einstein term and if $\lim_{R\to 0} f \sim R^2$

We have a class of solutions $b(r)=\frac{\exp[-\int dr\,h(r)]}{K+4\int\frac{dr\,a(r)\,\exp[-\int dr\,h(r)]}{r[a(r)+ra'(r)]}}$

In this case, the GR limit is not present!

Spherical symmetry of *f*-gravity with constant Ricci scalar

Whenever the gravitational potential $g_{tt}(t, r)$ is described by a separable functions and $g_{rr}(t, r)$ is time - independent, by the definition of the Ricci scalar, one gets that R = constant and, at the same time, the final solutions of the field equations will be static if the spherical symmetry is invoked: **Birkhoff theorem** is valid also for this particular class of *f*- gravity theories.

$$\begin{cases} R_{\mu\nu} + \lambda g_{\mu\nu} = q\,\mathcal{X}\,T_{\mu\nu} \\ \\ R_0 = q\,\mathcal{X}\,T - 4\lambda \end{cases}$$

$$ds^2 = \left(1 + \frac{k_1}{r} + \frac{q\mathcal{X}\rho - \lambda}{3} r^2\right) dt^2 - \frac{dr^2}{1 + \frac{k_1}{r} + \frac{q\mathcal{X}\rho - \lambda}{3} r^2} - r^2 d\Omega$$

In other words, any *f* – theory, in the case of constant curvature scalar, exhibits solutions with cosmological constant as the Schwarzschild - de Sitter solution.

Spherical symmetry of *f*-gravity with constant Ricci scalar

In standard GR- gravity, the constant curvature solutions different from zero are obtained only in presence of matter because of the proportionality of the Ricci scalar to the trace of the energy-momentum tensor of matter, or, on the other side, one can get a similar situation in presence of the cosmological constant. Actually the big difference between GR and higher order gravity is that, **the Schwarzschild-de Sitter solution is not necessarily given by the cosmological term** while the effect of an "effective" cosmological constant, in the low energy limit, can be played by the higher order derivative contributions evaluated on Ricci constant backgrounds.

Summary of solutions (with constant value of Ricci scalar)

f - theory:		Field equations:
R	$\longrightarrow$	$R_{\mu\nu}=0$, with $R=0$
$\xi_1 R+\xi_2 R^n$	$\longrightarrow$	$\begin{cases} R_{\mu\nu}=0 & \text{with } R=0,\ \ \xi_1\neq 0 \\ R_{\mu\nu}+\lambda g_{\mu\nu}=0 & \text{with } R=\left[\frac{\xi_1}{(n-2)\xi_2}\right]^{\frac{1}{n-1}},\ \ \xi_1\neq 0,\ \ n\neq 2 \\ 0=0 & \text{with } R=0,\ \ \xi_1=0 \\ R_{\mu\nu}+\lambda g_{\mu\nu}=0 & \text{with } R=R_0,\ \xi_1=0,\ n=2 \end{cases}$
$\xi_1 R+\xi_2 R^{-m}$	$\longrightarrow$	$R_{\mu\nu}+\lambda g_{\mu\nu}=0$ with $R=\left[-\frac{(m+2)\xi_2}{\xi_1}\right]^{\frac{1}{m+1}}$
$\xi_1 R+\xi_2 R^n+\xi_3 R^{-m}$	$\longrightarrow$	$R_{\mu\nu}+\lambda g_{\mu\nu}=0$, with $R=R_0$ so that $\xi_1 R_0^{m+1}+(2-n)\xi_2 R_0^{n+m}+(m+2)\xi_3=0$
$\frac{R}{\xi_1+R}$	$\longrightarrow$	$\begin{cases} R_{\mu\nu}=0 & \text{with } R=0 \\ R_{\mu\nu}+\lambda g_{\mu\nu}=0 & \text{with } R=-\frac{\xi_1}{2} \end{cases}$
$\frac{1}{\xi_1+R}$	$\longrightarrow$	$R_{\mu\nu}+\lambda g_{\mu\nu}=0$, with $R=-\frac{2\xi_1}{3}$

The Noether Symmetry approach to *f*-gravity

We worked out an approach to obtain time-independent spherically symmetric solutions in *f*-gravity. In order to develop such an approach, we need to deduce a point-like Lagrangian from the general action. Such a Lagrangian can be obtained by imposing the spherical symmetry in the field action . As a consequence, the infinite number of degrees of freedom of the original field theory will be reduced to a finite number.

$$\mathcal{L} = -\frac{A^{1/2} f_R}{2MB^{1/2}} M'^2 - \frac{f_R}{A^{1/2}B^{1/2}} A'M' - \frac{M f_{RR}}{A^{1/2}B^{1/2}} A'R' +$$

$$-\frac{2A^{1/2} f_{RR}}{B^{1/2}} R'M' - A^{1/2}B^{1/2}[(2+MR)f_R - Mf]$$

The Euler-Lagrange equations are in terms of the function *A, B, M, R*. The field equation for *R* corresponds to the constraint among the configuration coordinates.

It is worth noting that the Hessian determinant is zero! The point-like Lagrangian does not depend on *B*. In other words, *B* does not contribute to dynamics and its equation has to be considered as a further constraint equation.

The Noether Symmetry approach to *f*-gravity

The field-equations approach and the point-like Lagrangian approach differ since the symmetry, in our case the spherical one, can be imposed whether in the field equations, after standard variation with respect to the metric, or directly into the Lagrangian, which becomes point-like.

Field equations approach		Point-like Lagrangian approach
$\downarrow$		$\downarrow$
$\delta\int d^4x\sqrt{-g}f=0$	$\leftrightarrows$	$\delta\int dr\mathcal{L}=0$
$\downarrow$		$\downarrow$
$H_{\mu\nu}=\partial_\rho\frac{\partial(\sqrt{-g}f)}{\partial_\rho g^{\mu\nu}}-\frac{\partial(\sqrt{-g}f)}{\partial g^{\mu\nu}}=0$		$\frac{d}{dr}\nabla_{q'}\mathcal{L}-\nabla_q\mathcal{L}=0$
	$\leftrightarrows$	
$H=g^{\alpha\beta}H_{\alpha\beta}=0$		$E_{\mathcal{L}}=\underline{q}'\cdot\nabla_{q'}\mathcal{L}-\mathcal{L}$
$\downarrow$		$\downarrow$
$H_{tt}=0$	$\leftrightarrows$	$\frac{d}{dr}\frac{\partial\mathcal{L}}{\partial A'}-\frac{\partial\mathcal{L}}{\partial A}=0$
$H_{rr}=0$	$\leftrightarrows$	$\frac{d}{dr}\frac{\partial\mathcal{L}}{\partial B'}-\frac{\partial\mathcal{L}}{\partial B}\propto E_{\mathcal{L}}=0$
$H_{\theta\theta}=\csc^2\theta H_{\phi\phi}=0$	$\leftrightarrows$	$\frac{d}{dr}\frac{\partial\mathcal{L}}{\partial M'}-\frac{\partial\mathcal{L}}{\partial M}=0$
$H=A^{-1}H_{tt}-B^{-1}H_{rr}-2M^{-1}\csc^2\theta H_{\phi\phi}=0$	$\leftrightarrows$	A combination of the above equations

The Noether Symmetry approach to f-gravity

We find a relation between the metric potentials

$$B = \frac{2M^2 f_{RR} A' R' + 2M f_R A' M' + 4AM f_{RR} M' R' + A f_R M'^2}{2AM[(2 + MR) f_R - M f]}$$

If $f = R$ we find the usual condition for A and B. Now it is possible to reduce the configuration space of dynamics and we have

$$\mathbf{L} = \underline{q}'^{t}\,\hat{\mathbf{L}}\,\underline{q}' = \frac{[(2 + MR) f_R - f M]}{M}$$

$$\times [2M^2 f_{RR} A' R' + 2MM'(f_R A' + 2A f_{RR} R') + A f_R M'^2]$$

By applying the Noether Symmetry Approach we find a constant motion for a power law f

$$\Sigma_0 = \underline{\alpha} \cdot \nabla_{q'} \mathbf{L} =$$

$$= 2skMR^{2s-3}[2s + (s-1)MR][(s-2)RA' - (2s^2 - 3s + 1)AR']$$

The solution, for s = 5/2, is $ds^2 = \frac{1}{\sqrt{5}}(k_2 + k_1 r)dt^2 - \frac{1}{2}\left(\frac{1}{1 + \frac{k_2}{k_1 r}}\right)dr^2 - r^2 d\Omega$

Spherically symmetric solution with space-dependent scalar curvature for *f*-gravity by perturbations

If the Ricci scalar is only space-dependent we find as solution

$$a(r) = \frac{b(r)\, e^{-\frac{2}{3}\int \frac{[R+(2\mathcal{F}-R\mathcal{F}')]b(r)}{R'\mathcal{F}''}dr}}{r^4 R'^2 \mathcal{F}''^2}$$

$$f = R + \mathcal{F}(R), \text{ with } \mathcal{F}(R) \ll R$$

$$b(r) = -\frac{3(rR'\mathcal{F}'')_{,r}}{rR}$$

Also in this case, the Birkhoff theorem is satisfied.

The only off-diagonal component equation non identically vanishing is

$$\frac{d}{dr}\left(r^2 f'\right)\dot{g}_{rr}(t,r) = 0$$

Other choices are not possible: we would find incompatibilities.

A perturbative approach to *f*-gravity

If we suppose, for the metric, an expression $g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu}$ and compute the perturbations in the field equations, we obtain the equations:

$$f'^{(0)} R^{(0)}_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} f^{(0)} + \mathcal{H}^{(0)}_{\mu\nu} = \mathcal{X} T^{(0)}_{\mu\nu}$$

$$f'^{(0)} \left\{ R^{(1)}_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} R^{(1)} \right\} + f''^{(0)} R^{(1)} R^{(0)}_{\mu\nu} - \frac{1}{2} f^{(0)} g^{(1)}_{\mu\nu} + \mathcal{H}^{(1)}_{\mu\nu} = \mathcal{X} T^{(1)}_{\mu\nu}$$

Besides if we consider $f = R + \mathcal{F}(R)$ The set of equations is

$$R^{(0)}_{\mu\nu} - \frac{1}{2} R^{(0)} g^{(0)}_{\mu\nu} = G^{(0)}_{\mu\nu} = \mathcal{X} T^{(0)}_{\mu\nu} \qquad \text{(GR equations)}$$

$$R^{(1)}_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} R^{(1)} - \frac{1}{2} g^{(1)}_{\mu\nu} R^{(0)} - \frac{1}{2} g^{(0)}_{\mu\nu} \mathcal{F}^{(0)} + \mathcal{F}'^{(0)} R^{(0)}_{\mu\nu} + \mathcal{H}^{(1)}_{\mu\nu} = \mathcal{X} T^{(1)}_{\mu\nu}$$

Eddington parameters and *f*-theory by scalar-tensor theory analogy

By conformal transformations between the Jordan and Einstein frames, it is possible to red obtain a relation between the analytic function f, its derivatives and the PPN parameters.. We find the relations

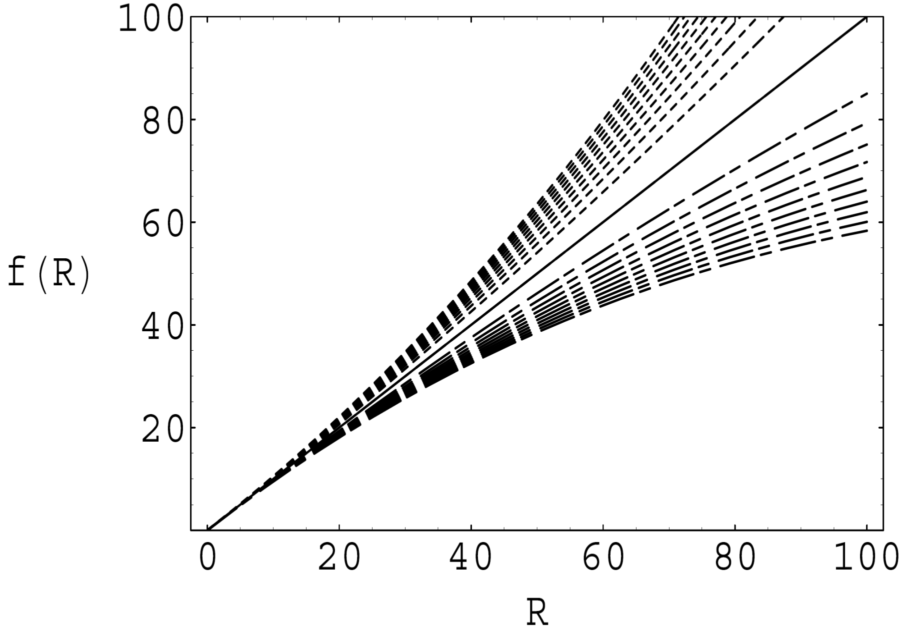
$$\gamma - 1 = -\frac{f''^2}{f' + 2f''^2}, \qquad \beta - 1 = \frac{1}{4}\left(\frac{f' \cdot f''}{2f' + 3f''^2}\right)\frac{d\gamma}{dR}$$

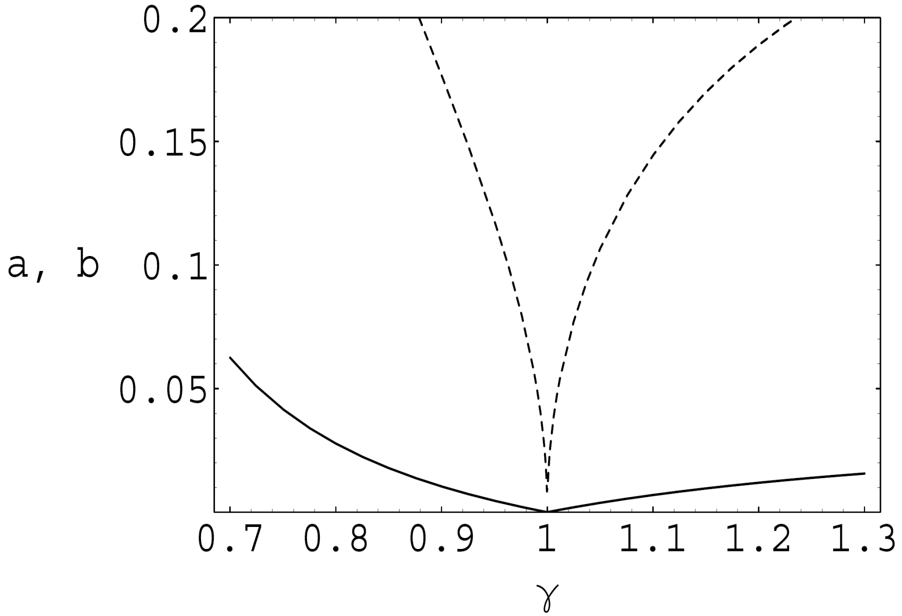
$$f_\pm = \frac{1}{12}\left|\frac{1-\gamma}{2\gamma-1}\right| R^3 \pm \frac{1}{2}\sqrt{\left|\frac{1-\gamma}{2\gamma-1}\right|} R^2 + R + \Lambda$$

Mercury Perihelion Shift	$\lvert 2\gamma - \beta - 1\rvert < 3 \times 10^{-3}$
Lunar Laser Ranging	$4\beta - \gamma - 3 = -(0.7 \pm 1) \times 10^{-3}$
Very Long Baseline Interf.	$\lvert\gamma - 1\rvert = 4 \times 10^{-4}$
Cassini Spacecraft	$\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$

Eddington parameters and *f*-theory by scalar-tensor theory analogy

The "solutions" for *f*-gravity are compatible with the Taylor expansion of *f*: the experimental value of Eddington's parameters determines the weight of the square and cubic term. Plot of the modulus of coefficients: *a* (R^3) (line) and *b* (R^2) (dashed line).





Newtonian limit of *f*-theory in vacuum with spherically symmetric metric (standard coordinates)

We start with a metric as follows

$$\begin{cases} g_{tt}(t,r) \simeq 1 + g_{tt}^{(2)}(t,r) + g_{tt}^{(4)}(t,r) \\ g_{rr}(t,r) \simeq -1 + g_{rr}^{(2)}(t,r) \\ g_{\theta\theta}(t,r) = -r^2 \\ g_{\phi\phi}(t,r) = -r^2 \sin^2\theta \end{cases}$$

Any *f* can be developed as Taylor series in some point

$$f = \sum_n \frac{f^n(R_0)}{n!}(R-R_0)^n \simeq f_0 + f_1 R + f_2 R^2 + f_3 R^3 + \ldots$$

$$\begin{cases} ds^2 = \left[1 - \frac{r_g}{f_1 r} + \frac{\delta_2(t)}{3\lambda}\frac{e^{-\lambda r}}{\lambda r}\right]dt^2 - \left[1 + \frac{r_g}{f_1 r} + \frac{\delta_2(t)}{3\lambda}\frac{\lambda r+1}{\lambda r}e^{-\lambda r}\right]dr^2 - r^2 d\Omega \\ R = \frac{\delta_2(t)e^{-\lambda r}}{r} \end{cases} \qquad \lambda \doteq \sqrt{-\frac{f_1}{6f_2}}$$

Newtonian limit of *f*-theory in vacuum with spherically symmetric metric (standard coordinates)

At third order we have

$$f_1 g^{(2)}_{rr,t} + 2 f_2 r R^{(2)}_{,tr} = 0$$

This equation states the relation between the "rr" component and the Ricci scalar. In fact, generally, if the "rr" component is time-independent and the "tt" component is the product of two functions (one of time and other one of the space) the Ricci scalar is time independent. Indeed exists every a redefinition of time gives back a metric time-independent. **Is not verified the Birkhoff theorem in f(R)?** It works only at Newtonian level.

Therefore, the Birkhoff theorem is not a general result for higher order gravity but, on the other hand, in the limit of small velocities and weak fields (which is enough to deal with the Solar System gravitational experiments), one can assume that the gravitational potential is effectively time-independent.

Newtonian limit of quadratic theory of gravity in presence of matter with spherically symmetric metric (isotropic coordinates)

Lagrangian: $$f = a_1 R + a_2 R^2 + a_3 R_{\alpha\beta}R^{\alpha\beta}$$

Metric: $$ds^2 = \Big[1 + 2\Phi\Big]dt^2 - \Big[1 - 2\Psi\Big]\delta_{ij}dx^i dx^j$$

Field equations:
$$\begin{cases} 2a_1\triangle\Psi - 2(4a_2 + a_3)\triangle^2\Psi + 2(2a_2 + a_3)\triangle^2\Phi = \mathcal{X}\rho\,, \\ \\ \triangle\Big[a_1(\Psi - \Phi) + (4a_2 + a_3)\triangle\Phi - (8a_2 + 3a_3)\triangle\Psi\Big]\delta_{ij} \\ \qquad -\Big[a_1(\Psi - \Phi) + (4a_2 + a_3)\triangle\Phi - (8a_2 + 3a_3)\triangle\Psi\Big]_{,ij} = 0 \end{cases}$$

A discussion of particular solutions

Newtonian limit of quadratic theory of gravity in presence of matter with spherically symmetric metric (isotropic coordinates)

If we consider a point-like source, the solution is

$$
\begin{aligned}
ds^2 \;=\; & \left[1-\frac{r_g}{a_1|\mathbf{x}|}\left(1-\frac{4}{3}e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}e^{-\lambda_2|\mathbf{x}|}\right)\right]dt^2+ \\
& -\left[1+\frac{r_g}{a_1|\mathbf{x}|}\left(1-\frac{2}{3}e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}e^{-\lambda_2|\mathbf{x}|}\right)\right]\delta_{ij}dx^i dx^j
\end{aligned}
$$

We find two Yukawa-like corrections to the Newtonian potential. **The limit of theory is the Newtonian mechanics**. As we shall see later, we have a massive mode of propagation.

$$
\lambda_1^2 \doteq -\frac{a_1}{a_3}\,, \qquad \lambda_2^2 \doteq \frac{a_1}{2(3a_2+a_3)}
$$

Newtonian limit of quadratic theory of gravity in presence of matter with spherically symmetric metric (isotropic coordinates)

If we consider a spherically symmetric source of matter, the solutions internal and external are:

$$\begin{cases} {}_A\Phi_{in}(\mathbf{x}) = -\frac{\sqrt{2}\,\pi^{3/2}}{a_1}\frac{GM}{\xi^3}\left[\frac{\lambda_1^2(2+3\lambda_2^2\xi^2)-8\lambda_2^2}{\lambda_1^2\lambda_2^2} - |\mathbf{x}|^2 + 8e^{-\lambda_1\xi}(1+\lambda_1\xi)\frac{\sinh(\lambda_1|\mathbf{x}|)}{\lambda_1^3|\mathbf{x}|}\right. \\ \qquad\left. -2e^{-\lambda_2\xi}(1+\lambda_2\xi)\frac{\sinh(\lambda_2|\mathbf{x}|)}{\lambda_2^3|\mathbf{x}|}\right] \\ {}_A\Phi_{out}(\mathbf{x}) = -\frac{2\sqrt{2}\,\pi^{3/2}}{a_1}\frac{GM}{|\mathbf{x}|} + \frac{8\sqrt{2}\,\pi^{3/2}}{a_1}\frac{GM}{\lambda_1^3\xi^3}[\lambda_1\xi\cosh(\lambda_1\xi) - \sinh(\lambda_1\xi)]\frac{e^{-\lambda_1|\mathbf{x}|}}{|\mathbf{x}|} \\ \qquad -\frac{2\sqrt{2}\,\pi^{3/2}}{a_1}\frac{GM}{\lambda_2^3\xi^3}[\lambda_2\xi\cosh(\lambda_2\xi) - \sinh(\lambda_2\xi)]\frac{e^{-\lambda_2|\mathbf{x}|}}{|\mathbf{x}|} \end{cases}$$

The limit of GR is recovered. Besides the limit of point-like mass is well defined.

The gravitational potential depends on the distribution of matter: the Gauss theorem is KO! Obviously …

Other solutions are possible: three independent Green functions exist

What is wrong in PPN-parameterization of Higher Order Gravity?

The Eddington parameters are defined by relations

$$ds^2 \simeq \left[1 - \alpha\frac{r_g}{r'} + \frac{\beta}{2}\left(\frac{r_g}{r'}\right)^2 + \dots\right]dt^2 - \left[1 + \gamma\frac{r_g}{r'} + \dots\right]\left[dr'^2 + r'^2 d\Omega\right]$$

… and our Newtonian limit admits another space-dependence. t**he parameterization by Eddington parameters could be misinterpreted.**

The Eddington idea born when Lagrangians proportional to Ricci scalar were investigated.

f-gravity and O'Hanlon theory

The f-gravity is analogous O' Hanlon theory and not to Brans-Dicke theory.

$$\begin{cases} \mathcal{A}^{f}_{JF} = \int d^4x\sqrt{-g}\Big[f + \mathcal{X}\mathcal{L}_m\Big] \\ \mathcal{A}^{BD}_{JF} = \int d^4x\sqrt{-g}\Big[\phi R - \omega_{BD}\frac{\phi_{;\alpha}\phi^{;\alpha}}{\phi} + \mathcal{X}\mathcal{L}_m\Big] \\ \mathcal{A}^{OH}_{JF} = \int d^4x\sqrt{-g}\Big[\phi R + V(\phi) + \mathcal{X}\mathcal{L}_m\Big] \end{cases}$$

$$\begin{cases} \phi = f' \\ V(\phi) = f - f'R \\ \phi\frac{dV(\phi)}{d\phi} - 2V(\phi) = f'R - 2f \end{cases}$$

If we perform the Newtonian limit of O' Hanlon theory and come back to f-theory approach the solution

$$\begin{cases} g_{tt} = 1 - \frac{2}{3a}\frac{r_g}{|\mathbf{x}|} - \sqrt{\frac{\pi}{2}}\frac{1}{3a}\frac{r_g e^{-\lambda|\mathbf{x}|}}{|\mathbf{x}|} \\ g_{ij} = -\left\{1 + \frac{1}{3a}\frac{r_g}{|\mathbf{x}|} - \sqrt{\frac{\pi}{2}}\frac{r_g}{3a}\left[\left(\frac{1}{|\mathbf{x}|} - \frac{2}{\lambda^2|\mathbf{x}|^3} - \frac{2}{\lambda|\mathbf{x}|^2}\right)e^{-\lambda|\mathbf{x}|} - \frac{2}{\lambda^2|\mathbf{x}|^3}\right]\right\}\delta_{ij} \\ \qquad + \frac{(2\pi)^{1/2}r_g}{3a}\left[\left(\frac{1}{|\mathbf{x}|} + \frac{3}{\lambda|\mathbf{x}|^2} + \frac{3}{\lambda^2|\mathbf{x}|^3}\right)e^{-\lambda|\mathbf{x}|} - \frac{3}{\lambda^2|\mathbf{x}|^3}\right]\frac{x_i x_j}{|\mathbf{x}|^2} \end{cases}$$

If we turn off the modification in the theory we dont find the GR. The same situation for BD theory!

Post-Minkowskian limit of f–gravity: gravitational waves in f-gravity

The post-Minkowskian limit is recovered if we consider **only the weak field** hypothesis. In this case the time derivative is of the same order of space derivative. The Laplacian, in the PPN formalism, is replaced by the d'Alembertian.
The work hypothesis:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

The field equations in the harmonic gauge become

$$\begin{cases} \Box_\eta \tilde{h}_{\mu\nu} + \frac{1}{3\lambda^2}(\eta_{\mu\nu}\Box_\eta - \partial^2_{\mu\nu})\Box_\eta \tilde{h} = -\frac{2\mathcal{X}}{f_1} T^{(0)}_{\mu\nu} \\ \\ \Box_\eta \tilde{h} + \frac{1}{\lambda^2}\Box^2_\eta \tilde{h} = -\frac{2\mathcal{X}}{f_1} T^{(0)} \end{cases}$$

The solution in term of Fourier transform is

$$h_{\mu\nu}(k) = \frac{2\mathcal{X}}{f_1}\frac{S^{(0)}_{\mu\nu}(k)}{k^2} - \frac{\mathcal{X}}{3f_1}\frac{k^2\eta_{\mu\nu} + 2k_\mu k_\nu}{k^2(k^2-\lambda^2)} S^{(0)}(k) \qquad \Box_\eta h_{\mu\nu}(x) = -\frac{2\mathcal{X}}{f_1}\left[S^{(0)}_{\mu\nu}(x) + \Sigma^\lambda_{\mu\nu}(x)\right]$$

Energy-momentum tensor of *f*-gravity

As in GR, we try to define a Energy-momentum tensor. In *f*-gravity it is possible to write

$$t^{\lambda}{}_{\alpha} = \quad f'\left\{\left[\frac{\partial R}{\partial g_{\rho\sigma,\lambda}} - \frac{1}{\sqrt{-g}}\partial_{\xi}\left(\sqrt{-g}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}\right)\right]g_{\rho\sigma,\alpha} + \frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\xi\alpha}\right\}$$

$$-f''R_{,\xi}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\alpha} - \delta^{\lambda}_{\alpha}f\,.$$

from which we have

$$[\sqrt{-g}(t^{\lambda}{}_{\alpha} + 2\mathcal{X}T^{\lambda}{}_{\alpha})]_{,\lambda} = 0$$

in terms of perturbation:

$$t^{\lambda}{}_{\alpha} \sim f'(0)t^{\lambda}_{\alpha|_{\mathrm{GR}}} + f''(0)\{(h^{\rho\sigma}{}_{,\rho\sigma} - \Box h)[h^{\lambda\xi}{}_{,\xi\alpha} - h^{,\lambda}{}_{\alpha} - \frac{1}{2}\delta^{\lambda}_{\alpha}(h^{\rho\sigma}{}_{,\rho\sigma} - \Box h)]$$

$$-h^{\rho\sigma}{}_{,\rho\sigma\xi}h^{\lambda\xi}{}_{,\alpha} + h^{\rho\sigma}{}_{,\rho\sigma}{}^{\lambda}h_{,\alpha} + h^{\lambda\xi}{}_{,\alpha}\Box h_{,\xi} - \Box h^{,\lambda}h_{,\alpha}\}\,,$$

Conclusions

We have considered the Taylor expansion up to the second order of a generic *f*-gravity: the solution found (time-time component) is corrected by a Yukawa-like term with respect to the Newtonian one ($f = R$). Also in the case of quadratic theory we find a Yukawa corrections.

A discussion on the non-validity of the Gauss theorem has been given. Furthermore, for spherically symmetric distributions of matter, we discussed the inner and the outer solutions. Furthermore, it has been shown that the *Birkhoff theorem* is not a general result for *f* - gravity.

From other hand it is possible also to calculate Newtonian limit of such theories with a redefinition of the degrees of freedom by some scalar field. Also in this case, we found a Yukawa-like correction to classic Newtonian potential. Nevertheless when we turn off the modification of Hilbert-Einstein Lagrangian we do not obtain the right Newtonian potential.

We have discussed the differences between the post-Newtonian and the post - Minkowskian limit in *f* - gravity. The main result of such an investigation is the presence of massive degrees of freedom in the spectrum of gravitational waves which are strictly related to the modifications occurring into the gravitational potential. This occurrence could constitute an interesting opportunity for the detection and investigation of gravitational waves. To do this it needs to generalize the energy-momentum tensor for a generic *f*-gravity.

Conclusions

We have discussed a general method to find out exact solutions in *f*-gravity when a spherically symmetric background is taken into account. In particular, we have searched for exact spherically symmetric solutions in *f*-gravity by asking for the existence of Noether symmetries.

Finally, we have constructed a perturbation approach in which we search for spherical solutions at the 0*th*-order and then we search for solutions at the first order. The scheme is iterative and could be, in principle, extended to any order in perturbations. The crucial request is to take into account *f* - theories which are Taylor expandable about some constant value of the curvature scalar.

Starting from Tensor-multi-scalar theory of gravity we can show how a polynomial Lagrangian in the Ricci scalar *R*, compatible with the PPN-limit, can be recovered in the framework of f.gravity. The approach is based on the formulation of the PPN-limit of such gravity models developed in analogy with scalar-tensor gravity.

Perspectives

Experimental testing of all results:

1) Galaxies rotation curves (Capodimonte Deep Survey)
2) Solar System experiments (GAIA, LATOR)
3) Effective Theories from Unification Schemes
4)GW experiments (VIRGO, LIGO)

List of papers

1. *Fourth-order gravity and experimental constraints on Eddington parameters* - Capozziello S., Stabile A., Troisi A., Modern Physics Letters A **21**, 2291 (2006)

2. *Spherically symmetric solutions in f(R)-gravity via Noether symmetry approach* - Capozziello S., Stabile A., Troisi A., Classical and Quantum Gravity **24**, 2153 (2007)

3. *Newtonian limit of f(R)-gravity* - Capozziello S., Stabile A., Troisi A., Physical Review D **76**, 104019 (2007);

4. *Spherical symmetry in f(R)-gravity* - Capozziello S., Stabile A., Troisi A., - accepted by Classical and Quantum Gravity (arXiv 0709.0891);

5. *General considerations on the Newtonian limit of fourth order gravity* - Capozziello S., Stabile A., submitted to Classical and Quantum Gravity;

6. *The post minkowskian approximation in f(R)-gravity: Gravitational waves in higher order gravity* - Capozziello S., Stabile A., Troisi A., (in preparation);

7. *A new general solution in the Newtonian Limit of f(R)-gravity* (Addenda) - Capozziello S., Stabile A., Troisi A., submitted to Physical Review D;

8. *Some remarks on the analogy between Brans-Dicke theory and f(R)-gravity in the weak field and small velocity limit* - Capozziello S., Stabile A., Troisi A., (in preparation).

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The Weak Field Limit of Higher Order Gravity

Arturo Stabile

PhD Dissertation

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