Limite di Campo Debole per una Gravitazione del Quarto Ordine

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Fourth Order Gravity (FOG)

From action principle $\mathcal{A}^f = \int d^4x\sqrt{-g}\left[f(X,Y,Z)+\mathcal{X}\mathcal{L}_m\right]$

where $X = R$ (Ricci scalar), $Y = R_{\alpha\beta}R^{\alpha\beta}$ (Ricci tensor square) and $Z = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ (Riemann square)

The field equations are

$$H_{\mu\nu} = f_X R_{\mu\nu} - \frac{f}{2}g_{\mu\nu} - f_{X;\mu\nu} + g_{\mu\nu}\Box f_X + 2f_Y R_\mu{}^\alpha R_{\alpha\nu} - 2[f_Y R^\alpha{}_{(\mu}]_{;\nu)\alpha} + \Box[f_Y R_{\mu\nu}] + [f_Y R_{\alpha\beta}]^{;\alpha\beta}g_{\mu\nu}$$

$$+2f_Z R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} - 4[f_Z R_\mu{}^{\alpha\beta}{}_\nu]_{;\alpha\beta} = \mathcal{X}T_{\mu\nu}$$

and the trace is given by

$$H = g^{\alpha\beta}H_{\alpha\beta} = f_X R + 2f_Y R_{\alpha\beta}R^{\alpha\beta} + 2f_Z R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 2f + \Box[3f_X + f_Y R] + 2[(f_Y + 2f_Z)R^{\alpha\beta}]_{;\alpha\beta} = \mathcal{X}T$$

Why we have a theory of fourth order?

In GR theory we have naturally four divergences … while in FOG we must make them, but …

Weak field limit of FOG

There are two weak field limits:

- **Newtonian and Post Newtonian limit**
- **Post Minkowskian limit → Gravitational waves**

If one takes into account a system of gravitationally interacting massive particles, the kinetic energy will be, roughly, of the same order of magnitude as the typical potential energy. As a consequence

$$\bar{v}^2 \sim \frac{G\bar{M}}{\bar{r}}$$

The Newtonian and the Post-Newtonian approximation can be described as a method for obtaining the motion of the system to higher approximations than the first order[4] with respect to the quantities velocity square and / or gravitational potential assumed small with respect to the squared light speed.

This approximation is sometimes referred to as an expansion in inverse powers of the light speed.

Newtonian and Post-Newtonian Limit of FOG

First consequences are $\frac{\partial}{\partial t} \sim \mathbf{v} \cdot \nabla$ and $\frac{|\partial/\partial t|}{|\nabla|} \sim \mathcal{O}(1)$

The geodesic equation $\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\sigma\tau} \frac{dx^\sigma}{ds} \frac{dx^\tau}{ds} = 0$ can be rewritten

$$\frac{d^2 x^i}{dt^2} = -\Gamma^i_{tt} - 2\Gamma^i_{tm} \frac{dx^m}{dt} - \Gamma^i_{mn} \frac{dx^m}{dt} \frac{dx^n}{dt} + \left[\Gamma^t_{tt} + 2\Gamma^t_{tm} \frac{dx^m}{dt} + 2\Gamma^t_{mn} \frac{dx^m}{dt} \frac{dx^n}{dt}\right] \frac{dx^i}{dt}$$

Then at lowest level we have the Newtonian Mechanics $\frac{d^2 x^i}{dt^2} \simeq -\Gamma^i_{tt} \simeq -\frac{1}{2} \frac{\partial g_{tt}}{\partial x^i}$

The metric tensor becomes

$$\begin{cases} g_{tt}(t, \mathbf{x}) \simeq 1 + g^{(2)}_{tt}(t, \mathbf{x}) + g^{(4)}_{tt}(t, \mathbf{x}) + \mathcal{O}(6) \\ g_{ti}(t, \mathbf{x}) \simeq g^{(3)}_{ti}(t, \mathbf{x}) + \mathcal{O}(5) \\ g_{ij}(t, \mathbf{x}) \simeq -\delta_{ij} + g^{(2)}_{ij}(t, \mathbf{x}) + \mathcal{O}(4) \end{cases}$$

Newtonian of *f(R)*-Gravity

We start with a metric as follows

$$\begin{cases} g_{tt}(t,r) \simeq 1 + g^{(2)}_{tt}(t,r) + g^{(4)}_{tt}(t,r) \\ g_{rr}(t,r) \simeq -1 + g^{(2)}_{rr}(t,r) \\ g_{\theta\theta}(t,r) = -r^2 \\ g_{\phi\phi}(t,r) = -r^2 \sin^2\theta \end{cases}$$

Any *f(R)* can be developed as Taylor series in some point

$$f = \sum_n \frac{f^n(R_0)}{n!}(R-R_0)^n \simeq f_0 + f_1 R + f_2 R^2 + f_3 R^3 + \ldots$$

$$\begin{cases} f_1 r R^{(2)} - 2 f_1 g^{(2)}_{tt,r} + 8 f_2 R^{(2)}_{,r} - f_1 r g^{(2)}_{tt,rr} + 4 f_2 r R^{(2)} = 0 \\ f_1 r R^{(2)} - 2 f_1 g^{(2)}_{rr,r} + 8 f_2 R^{(2)}_{,r} - f_1 r g^{(2)}_{tt,rr} = 0 \\ 2 f_1 g^{(2)}_{rr} - r[f_1 r R^{(2)} - f_1 g^{(2)}_{tt,r} - f_1 g^{(2)}_{rr,r} + 4 f_2 R^{(2)}_{,r} + 4 f_2 r R^{(2)}_{,rr}] = 0 \\ f_1 r R^{(2)} + 6 f_2 [2 R^{(2)}_{,r} + r R^{(2)}_{,rr}] = 0 \\ 2 g^{(2)}_{rr} + r[2 g^{(2)}_{tt,r} - r R^{(2)} + 2 g^{(2)}_{rr,r} + r g^{(2)}_{tt,rr}] = 0 \end{cases}$$

If the Lagrangian is expanded around a vanishing value of the Ricci scalar (*R_0 = 0*), the field equation at lowest order implies that the cosmological constant contribution has to be zero whatever is the *f(R)*-Gravity model

$$f_0 = 0$$

Newtonian of *f(R)*-Gravity

The solution of metric tensor is

$$\begin{cases} ds^2 = \left[1 - \frac{r_g}{f_1 r} + \frac{\delta_2(t)}{3m}\frac{e^{-mr}}{mr}\right]dt^2 - \left[1 + \frac{r_g}{f_1 r} + \frac{\delta_2(t)}{3m}\frac{mr+1}{mr}e^{-mr}\right]dr^2 - r^2 d\Omega \\ R = \frac{\delta_2(t) e^{-mr}}{r} \end{cases} \qquad m \doteq \sqrt{-\frac{f_1}{6f_2}}$$

At third order the field equation states the relation between the "rr" component and the Ricci scalar

$$f_1 g^{(2)}_{rr,t} + 2f_2 r R^{(2)}_{,tr} = 0$$

In fact, generally, if the "rr" component is time-independent and the "tt" component is the product of two functions (one of time and other one of the space) the Ricci scalar is time independent. Indeed exists every a redefinition of time gives back a metric time-independent. **Is not verified the Birkhoff theorem in f(R)?** It works only at Newtonian level.

Therefore, the Birkhoff theorem is not a general result for higher order gravity but, on the other hand, in the limit of small velocities and weak fields (which is enough to deal with the Solar System gravitational experiments), one can assume that the gravitational potential is effectively time-independent.

Newtonian and Post-Newtonian of *f(R)*-Gravity in isotropic coordinates

The metric tensor is developed as

$$g_{\mu\nu} \sim \begin{pmatrix} 1 + g^{(2)}_{tt}(t,\mathbf{x}) + g^{(4)}_{tt}(t,\mathbf{x}) + \ldots & g^{(3)}_{ti}(t,\mathbf{x}) + \ldots \\ g^{(3)}_{ti}(t,\mathbf{x}) + \ldots & -\delta_{ij} + g^{(2)}_{ij}(t,\mathbf{x}) + \ldots \end{pmatrix}$$

The field equations at "Newtonian level" are

$$\begin{cases} H^{(2)}_{tt} = f'(0)R^{(2)}_{tt} - \frac{f'(0)}{2}R^{(2)} - f''(0)\triangle R^{(2)} = \mathcal{X}T^{(0)}_{tt} \\ H^{(2)}_{ij} = f'(0)R^{(2)}_{ij} + \left[\frac{f'(0)}{2}R^{(2)} + f''(0)\triangle R^{(2)}\right]\delta_{ij} - f''(0)R^{(2)}_{,ij} = 0 \\ H^{(2)} = -3f''(0)\triangle R^{(2)} - f'(0)R^{(2)} = \mathcal{X}T^{(0)} \end{cases}$$

At third order

$$H^{(3)}_{ti} = f'(0)R^{(3)}_{ti} - f''(0)R^{(2)}{}_{,ti} = \mathcal{X}T^{(1)}_{ti}$$

And at fourth order …

Newtonian and Post-Newtonian of *f(R)*-Gravity in isotropic coordinates

$$\left\{\begin{array}{l}
H_{tt}^{(4)} = f'(0)R_{tt}^{(4)} + f''(0)R^{(2)}R_{tt}^{(2)} - \frac{f'(0)}{2}R^{(4)} - \frac{f'(0)}{2}g_{tt}^{(2)}R^{(2)} - \frac{f''(0)}{4}{R^{(2)}}^2 \\
\quad -f''(0)\left[g_{mn,m}^{(2)}R^{(2)}{}_{,n} + \triangle R^{(4)} + g_{tt}^{(2)}\triangle R^{(2)} + g_{mn}^{(2)}R^{(2)}{}_{,mn} - \frac{1}{2}\nabla g_{mm}^{(2)}\cdot\nabla R^{(2)}\right] \\
\quad -f'''(0)\left[|\nabla R^{(2)}|^2 + R^{(2)}\triangle R^{(2)}\right] = \mathcal{X}T_{tt}^{(2)} \\
\\
H^{(4)} = -3f''(0)\triangle R^{(4)} - f'(0)R^{(4)} - 3f'''(0)\left[|\nabla R^{(2)}|^2 + R^{(2)}\triangle R^{(2)}\right] \\
\quad +3f''(0)\left[R_{,tt}^{(2)} - g_{mn}^{(2)}R_{,mn}^{(2)} - \frac{1}{2}\nabla(g_{tt}^{(2)} - g_{mm}^{(2)})\cdot\nabla R^{(2)} - g_{mn,m}^{(2)}R_{,n}^{(2)}\right] = \mathcal{X}T^{(2)}
\end{array}\right.$$

Newtonian and Post-Newtonian of *f(R)*-Gravity in isotropic coordinates

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The solutions can be formulated by using th Green functions method (the Newtonian level is linear!!!). The Ricci scalar is

$$R^{(2)}(t,\mathbf{x}) = \frac{m^2\mathcal{X}}{f'(0)}\int d^3\mathbf{x}'\mathcal{G}(\mathbf{x},\mathbf{x}')T^{(0)}(t,\mathbf{x}') \qquad m^2 \doteq -\frac{f'(0)}{3f''(0)}$$

where G(x,x’) is the Green function of field operator Δ - m^2.
The metric components are

$$g^{(2)}_{tt}(t,\mathbf{x}) = -\frac{\mathcal{X}}{2\pi f'(0)}\int d^3\mathbf{x}'\frac{T^{(0)}_{tt}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{4\pi}\int d^3\mathbf{x}'\frac{R^{(2)}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \frac{2}{3m^2}R^{(2)}(t,\mathbf{x})$$

$$g^{(3)}_{ti}(t,\mathbf{x}) = -\frac{\mathcal{X}}{2\pi f'(0)}\int d^3\mathbf{x}'\frac{T^{(1)}_{ti}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{1}{6\pi m^2}\frac{\partial}{\partial t}\int d^3\mathbf{x}'\frac{\nabla_{i'}R^{(2)}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

$$g^{(2)}_{ij}(t,\mathbf{x}) = \left[\frac{1}{4\pi}\int d^3\mathbf{x}'\frac{R^{(2)}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{2}{3m^2}R^{(2)}(t,\mathbf{x}) - \frac{1}{6\pi m^2}\frac{1}{|\mathbf{x}|^3}\int_{\Omega_{|\mathbf{x}|}} d^3\mathbf{x}'R^{(2)}(t,\mathbf{x}')\right]\delta_{ij}$$
$$+\left[\frac{1}{2\pi m^2|\mathbf{x}|^3}\int_{\Omega_{|\mathbf{x}|}} d^3\mathbf{x}'R^{(2)}(t,\mathbf{x}') - \frac{2}{3m^2}R^{(2)}(t,\mathbf{x})\right]\frac{x_ix_j}{|\mathbf{x}|^2}$$

And at fourth order …

Newtonian and Post-Newtonian of *f(R)*-Gravity in isotropic coordinates

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$$R^{(4)}(t,\mathbf{x}) = \int d^3\mathbf{x}'\mathcal{G}(\mathbf{x},\mathbf{x}')\left\{\frac{m^2\mathcal{X}}{f'(0)}T^{(2)}(t,\mathbf{x}') - g^{(2)}_{mn,m}(t,\mathbf{x}')R^{(2)}_{,n}(t,\mathbf{x}') - g^{(2)}_{mn}(t,\mathbf{x}')R^{(2)}_{,mn}(t,\mathbf{x}') \right.$$

$$+R^{(2)}_{,tt}(t,\mathbf{x}') - \frac{m^2}{\mu^4}\left[|\nabla_{\mathbf{x}'}R^{(2)}(t,\mathbf{x}')|^2 + R^{(2)}(t,\mathbf{x}')\triangle_{\mathbf{x}'}R^{(2)}(t,\mathbf{x}')\right]$$

$$\left. -\frac{1}{2}\nabla_{\mathbf{x}'}\left[g^{(2)}_{tt}(t,\mathbf{x}') - g^{(2)}_{mm}(t,\mathbf{x}')\right]\cdot\nabla_{\mathbf{x}'}R^{(2)}(t,\mathbf{x}')\right\}$$

$$\mu^4 \doteq -\frac{f'(0)}{3f'''(0)}$$

$$g^{(4)}_{tt}(t,\mathbf{x}) = \int d^3\mathbf{x}'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\left\{-\frac{\mathcal{X}T^{(2)}_{tt}(t,\mathbf{x}')}{2\pi f'(0)} + \frac{1}{6\pi\mu^4}\left[|\nabla R^{(2)}(t,\mathbf{x}')|^2 + R^{(2)}(t,\mathbf{x}')\triangle R^{(2)}(t,\mathbf{x}')\right]\right.$$

$$+\frac{1}{4\pi}\left[g^{(2)}_{mn}(t,\mathbf{x}')g^{(2)}_{tt,mn}(t,\mathbf{x}') - g^{(2)}_{tt,tt}(t,\mathbf{x}') - |\nabla_{\mathbf{x}'}g^{(2)}_{tt}(t,\mathbf{x}')|^2 - R^{(4)}(t,\mathbf{x}') - g^{(2)}_{tt}(t,\mathbf{x}')R^{(2)}(t,\mathbf{x}')\right]$$

$$+\frac{1}{6\pi m^2}\left[\frac{{R^{(2)}}^2(t,\mathbf{x}')}{4} - \frac{R^{(2)}(t,\mathbf{x}')\triangle g^{(2)}_{tt}(t,\mathbf{x}')}{2} + g^{(2)}_{mn,m}(t,\mathbf{x}')R^{(2)}{}_{,n}(t,\mathbf{x}') + \triangle R^{(4)}(t,\mathbf{x}')\right.$$

$$\left.\left. +g^{(2)}_{tt}(t,\mathbf{x}')\triangle R^{(2)}(t,\mathbf{x}') + g^{(2)}_{mn}(t,\mathbf{x}')R^{(2)}{}_{,mn}(t,\mathbf{x}') - \frac{1}{2}\nabla g^{(2)}_{mm}(t,\mathbf{x}')\cdot\nabla R^{(2)}(t,\mathbf{x}')\right]\right\}$$

Newtonian and Post-Newtonian of *f(R)*-Gravity in isotropic coordinates

The possible choices of the Green functions for spherically symmetric systems are the following

$$\mathcal{G}(\mathrm{x},\mathrm{x}') = \begin{cases} -\frac{1}{4\pi}\frac{e^{-m|\mathrm{x}-\mathrm{x}'|}}{|\mathrm{x}-\mathrm{x}'|} & \text{if} \quad m^2 > 0 \\ C_1\frac{e^{-im|\mathrm{x}-\mathrm{x}'|}}{|\mathrm{x}-\mathrm{x}'|} + C_2\frac{e^{im|\mathrm{x}-\mathrm{x}'|}}{|\mathrm{x}-\mathrm{x}'|} & \text{if} \quad m^2 < 0 \end{cases}$$

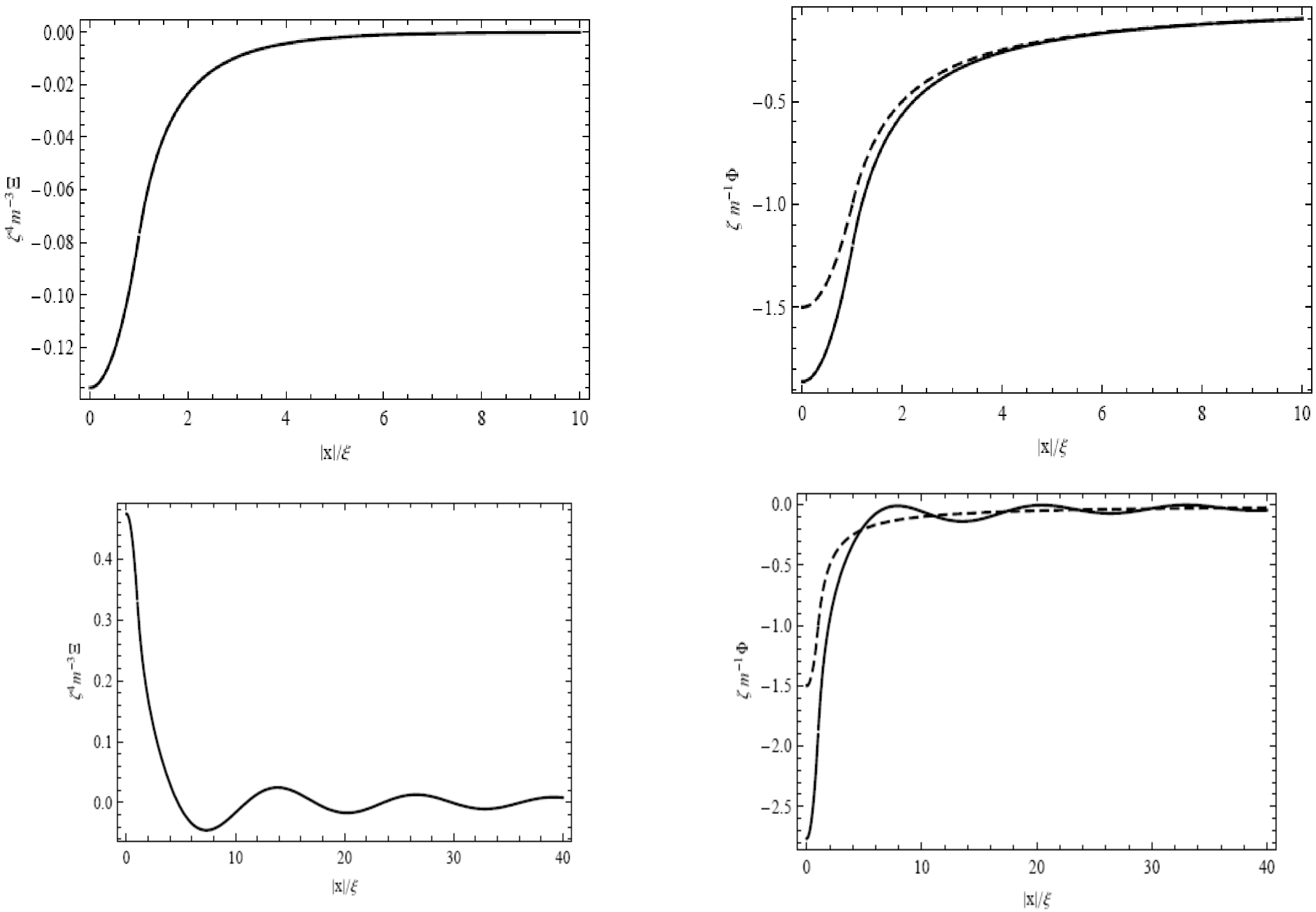
In the Newtonian limit of GR, the equation for the gravitational potential, generated by a point-like source is

$$\triangle_{\mathrm{x}}\mathcal{G}_{New.mech.}(\mathrm{x}-\mathrm{x}') = -4\pi\delta(\mathrm{x}-\mathrm{x}') \qquad \mathrm{g}_{New.mech.} = -\frac{GM(\mathrm{x}-\mathrm{x}')}{|\mathrm{x}-\mathrm{x}'|^3} = -GM\nabla_{\mathrm{x}}\mathcal{G}_{New.mech.}(\mathrm{x}-\mathrm{x}')$$

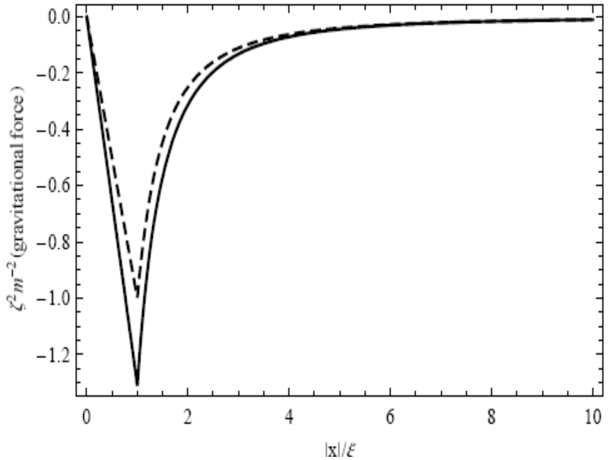
And the Gauss theorem holds $\displaystyle\int_\Sigma d\Sigma \ \ \mathrm{g}_{New.mech.}\cdot\hat{n} \propto M$

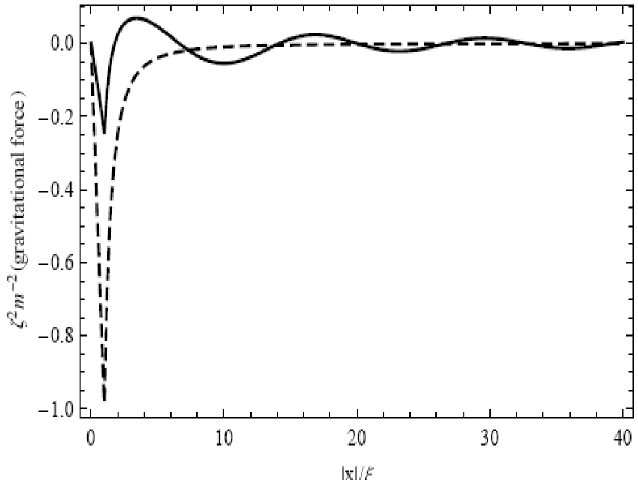
But in f(R)-Gravity $\displaystyle\int_\Sigma d\Sigma \ \ \mathrm{g}_{New.mech.}\cdot\hat{n} \propto M_\Sigma$ **Then the Gauss theorem is not verified!!!**

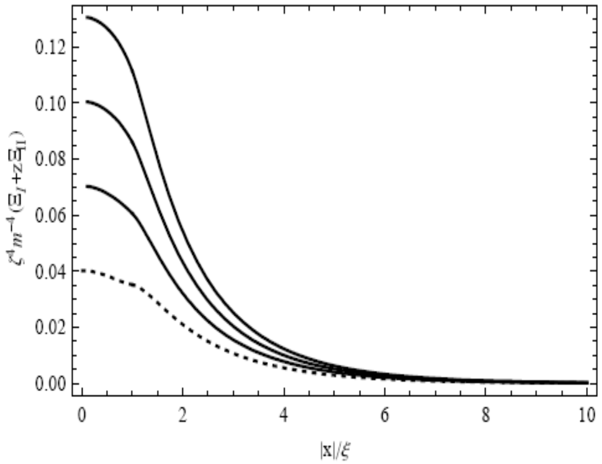
The potential and the Ricci scalar generated by ball-like source

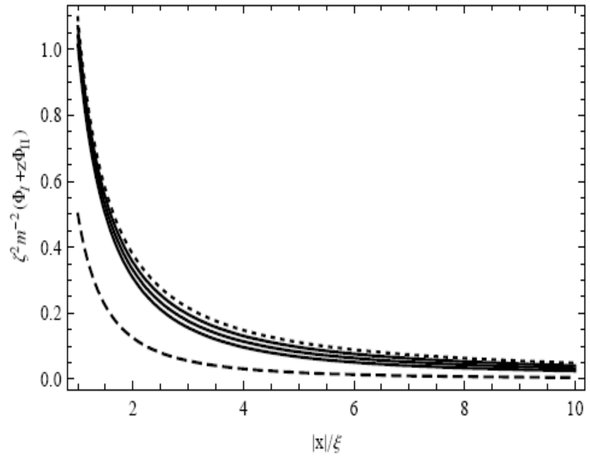


The potential and the Ricci scalar generated by ball-like source









The Newtonian limit of FOG

The field equations are

$$\begin{cases} (\triangle - m_2{}^2)R_{tt}^{(2)} + \left[\frac{m_2{}^2}{2} - \frac{m_1{}^2+2m_2{}^2}{6m_1{}^2}\triangle\right]X^{(2)} = -\frac{m_2{}^2\mathcal{X}}{f_X(0)}T_{tt}^{(0)} \\ (\triangle - m_2{}^2)R_{ij}^{(2)} + \left[\frac{m_1{}^2-m_2{}^2}{3m_1{}^2}\partial_{ij}^2 - \left(\frac{m_2{}^2}{2} - \frac{m_1{}^2+2m_2{}^2}{6m_1{}^2}\triangle\right)\delta_{ij}\right]X^{(2)} = 0 \\ (\triangle - m_1{}^2)X^{(2)} = \frac{m_1{}^2\mathcal{X}}{f_X(0)}T^{(0)} \end{cases}$$

$$\begin{cases} m_1{}^2 \doteq -\frac{f_X(0)}{3f_{XX}(0)+2f_Y(0)+2f_Z(0)} \\ m_2{}^2 \doteq \frac{f_X(0)}{f_Y(0)+4f_Z(0)]} \end{cases}$$

and the general solutions are $$X^{(2)}(t,\mathrm{x}) = \frac{m_1{}^2\mathcal{X}}{f_X(0)}\int d^3\mathrm{x}'\mathcal{G}_1(\mathrm{x},\mathrm{x}')T^{(0)}(t,\mathrm{x}')$$

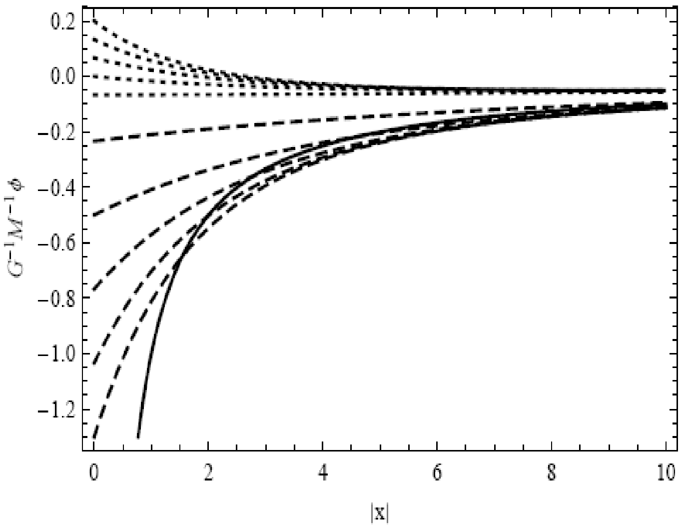
$$g_{tt}^{(2)}(t,\mathrm{x}) = \frac{1}{2\pi}\int d^3\mathrm{x}'d^3\mathrm{x}''\frac{\mathcal{G}_2(\mathrm{x}',\mathrm{x}'')}{|\mathrm{x}-\mathrm{x}'|}\left[\frac{m_2{}^2\mathcal{X}}{f_X(0)}T_{tt}^{(0)}(t,\mathrm{x}'') - \frac{(m_1{}^2+2m_2{}^2)\mathcal{X}}{6f_X(0)}T^{(0)}t,\mathrm{x}'') + \frac{m_2{}^2-m_1{}^2}{6}X^{(2)}(t,\mathrm{x}'')\right]$$

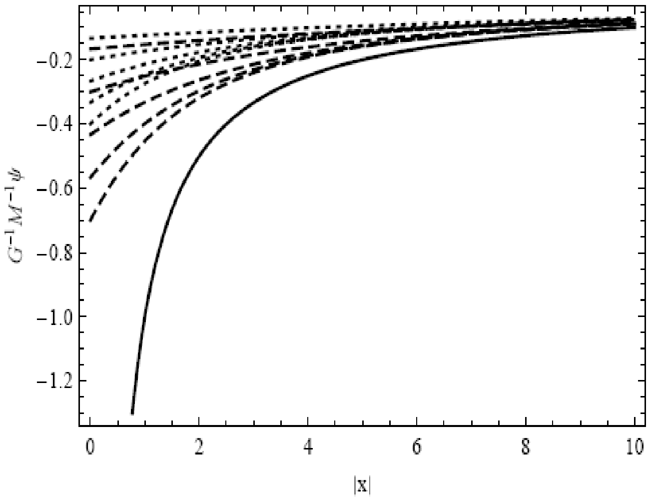
$$g_{ij}^{(2)} = 2\,\Psi\,\delta_{ij} = -\frac{\delta_{ij}}{2\pi}\int d^3\mathrm{x}'d^3\mathrm{x}''\frac{\mathcal{G}_2(\mathrm{x}',\mathrm{x}'')}{|\mathrm{x}-\mathrm{x}'|}\left(\frac{m_2{}^2}{2} - \frac{m_1{}^2+2m_2{}^2}{6m_1{}^2}\triangle_{\mathrm{x}''}\right)X^{(2)}(\mathrm{x}'')$$

The solutions at Newtonian limit of FOG

In the case of pointlike source of mass *M*

$$\begin{cases} X^{(2)} = -\dfrac{r_g\, {m_1}^2}{f_X(0)}\dfrac{e^{-m_1|\mathbf{x}|}}{|\mathbf{x}|} \\[2ex] \Phi = -\dfrac{GM}{f_X(0)}\left[\dfrac{1}{|\mathbf{x}|}+\dfrac{1}{3}\dfrac{e^{-m_1|\mathbf{x}|}}{|\mathbf{x}|}-\dfrac{4}{3}\dfrac{e^{-m_2|\mathbf{x}|}}{|\mathbf{x}|}\right] \\[2ex] \Psi = -\dfrac{GM}{f_X(0)}\left[\dfrac{1}{|\mathbf{x}|}-\dfrac{1}{3}\dfrac{e^{-m_1|\mathbf{x}|}}{|\mathbf{x}|}-\dfrac{2}{3}\dfrac{e^{-m_2|\mathbf{x}|}}{|\mathbf{x}|}\right] \end{cases}$$





For a physical interpretation of potential we must have

$$f_{XX}(0)+f_Y(0)+2f_Z(0) < 0$$

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