Axially symmetric solutions in f(R)-gravity

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Summary

- Spherical symmetry in f(R)-gravity
- Noether symmetry approach and spherical symmetry
- Axial symmetry derived from spherical symmetry
- Axially symmetric solutions in f(R)-gravity
- Physical applications
- Conclusions

Spherical symmetry in f(R)-gravity

Let us consider an analytic function f(R) .The variational principle for this action is

$$\boxed{\delta \int \mathrm{d}^4 x \sqrt{-g}[f(R) + \mathcal{X}\mathcal{L}_m] = 0,}$$

By varying with respect to the metric, we obtain the field equations

$$\boxed{\begin{cases} H_{\mu\nu} = f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - f'(R)_{;\mu\nu} + g_{\mu\nu} \Box f'(R) = \mathcal{X} T_{\mu\nu} \\ H = g^{\rho\sigma} H_{\rho\sigma} = 3 \Box f'(R) + f'(R) R - 2 f(R) = \mathcal{X} T, \end{cases}}$$

The most general spherically symmetric solution can be written as

f

$$\boxed{\mathrm{d}s^2 = m_1(t', r')\, \mathrm{d}t'^2 + m_2(t', r')\, \mathrm{d}r'^2 + m_3(t', r')\, \mathrm{d}t'\, \mathrm{d}r' + m_4(t', r')\, \mathrm{d}\Omega,}$$

Spherical symmetry in f(R)-gravity

We can consider a coordinate transformation that maps metric in a new one where the off-diagonal term vanishes and $m_4(t', r') = -r^2$, that is,

$$ds^2 = g_{tt}(t,r)\,dt^2 - g_{rr}(t,r)\,dr^2 - r^2\,d\Omega.$$

by inserting this metric into the field equations , one obtains

$$\begin{cases} f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + \mathcal{H}_{\mu\nu} = \mathcal{X}T_{\mu\nu} \\ f'(R)R - 2f(R) + \mathcal{H} = \mathcal{X}T, \end{cases}$$

Spherical symmetry in f(R)-gravity

...where the two quantities $\mathcal{H}_{\mu\nu}$ and $\mathcal{H}$ read

$$\mathcal{H}_{\mu\nu} = -f''(R)\Big\{R_{,\mu\nu} - \Gamma^{t}_{\mu\nu}R_{,t} - \Gamma^{r}_{\mu\nu}R_{,r} - g_{\mu\nu}\Big[\big({g^{tt}}_{,t} + g^{tt}(\ln\sqrt{-g})_{,t}\big)R_{,t} + \big({g^{rr}}_{,r} + g^{rr}(\ln\sqrt{-g})_{,r}\big)R_{,r} + g^{tt}R_{,tt} + g^{rr}R_{,rr}\Big]\Big\} - f'''(R)\Big[R_{,\mu}R_{,\nu} - g_{\mu\nu}\big(g^{tt}{R_{,t}}^{2} + g^{rr}{R_{,r}}^{2}\big)\Big]$$

$$\mathcal{H} = g^{\sigma\tau}\mathcal{H}_{\sigma\tau} = 3f''(R)\Big[\big({g^{tt}}_{,t} + g^{tt}(\ln\sqrt{-g})_{,t}\big)R_{,t} + \big({g^{rr}}_{,r} + g^{rr}(\ln\sqrt{-g})_{,r}\big)R_{,r} + g^{tt}R_{,tt} + g^{rr}R_{,rr}\Big] + 3f'''(R)\Big[g^{tt}{R_{,t}}^{2} + g^{rr}{R_{,r}}^{2}\Big].$$

Now our task is to find out the exact spherically symmetric solutions.

Spherical symmetry in f(R)-gravity

In the case of the time-independent metric, i.e $g_{tt} = a(r)$ *and* $g_{rr} = b(r)$*, the Ricci scalar can be recast as a Bernoulli equation of index 2 with respect to the metric potential b(r)*

$$b'(r) + \left\{ \frac{r^2 a'(r)^2 - 4a(r)^2 - 2ra(r)[2a(r)' + ra(r)'']}{ra(r)[4a(r) + ra'(r)]} \right\} b(r) + \left\{ \frac{2a(r)}{r} \left[\frac{2 + r^2 R(r)}{4a(r) + ra'(r)} \right] \right\} b(r)^2 = 0,$$

A general solution is

$$b(r) = \frac{\exp\left[-\int \mathrm{d}r\, h(r)\right]}{K + \int \mathrm{d}r l(r) \exp\left[-\int \mathrm{d}r h(r)\right]},$$

Capozziello S,Stabile A and Troisi A, 2008 CQG 25, 085004

Spherical symmetry in f(R)-gravity

We fix $l(r) = 0$; ⟶ *solutions with a Ricci scalar scaling as* $\boxed{-\frac{2}{r^2}}$ *in terms of the radial coordinate.*

It is not possible to have $h(r) = 0$ ⟶ *imaginary solutions.*

Particular consideration deserves the limit $\boxed{r \rightarrow \infty}$

To achieve a gravitational potential $b(r)$ with the correct Minkowski limit, both $h(r)$ and $l(r)$ have to go to zero at infinity, provided that the quantity $r^2R(r)$ turns out to be constant;

⬇

$b'(r) = 0$, and finally also the metric potential $b(r)$ has a correct Minkowski limit.

Spherical symmetry in f(R)-gravity

In general, if we ask for the asymptotic flatness of the metric as a feature of the theory, the Ricci scalar has to evolve to infinity as r^{-n} with $n \geq 2$. Formally, it has to be

$$\boxed{\lim_{r\to\infty} r^2 R(r) = r^{-n}}$$

with $n \in \mathcal{N}$. Any other behavior of the Ricci scalar could affect the requirement to achieve a correct asymptotic flatness.

Noether symmetry approach and spherical symmetry

Spherically symmetric solutions can be achieved by starting from a point-like f (R)- Lagrangian. Such a Lagrangian can be directly obtained by imposing the spherical symmetry in action .

$$\boxed{\mathrm{d}s^2 = A(r)\,\mathrm{d}t^2 - B(r)\,\mathrm{d}r^2 - M(r)\,\mathrm{d}\Omega\,,}$$

and then the point-like f (R)-Lagrangian reads

$$\boxed{\begin{aligned}\mathcal{L} = &-\frac{A^{1/2}f'}{2MB^{1/2}}M'^2 - \frac{f'}{A^{1/2}B^{1/2}}A'M' - \frac{Mf''}{A^{1/2}B^{1/2}}A'R' \\ &- \frac{2A^{1/2}f''}{B^{1/2}}R'M' - A^{1/2}B^{1/2}[(2+MR)f' - Mf],\end{aligned}}$$

Capozziello S, Stabile A and Troisi A 2007 CQG 24 2153

Noether symmetry approach and spherical symmetry

A point transformation $Q^i = Q^i(q)$ can depend on one (or more than one) parameter.

Assume that a point transformation depends on a parameter ε, i.e. $Q^i = Q^i(q, \varepsilon)$, and that it gives rise to a one-parameter Lie group.

For infinitesimal values of ε, the transformation is then generated by a vector field: for instance, $\partial/\partial x$ represents a translation along the x axis, $x(\partial/\partial y) - y(\partial/\partial x)$ is a rotation around the z axis and so on.

An infinitesimal point transformation is represented by a generic vector field on Q

$$\boxed{\mathbf{X} = \alpha^i(\mathbf{q})\frac{\partial}{\partial q^i}.}$$

Vector field

The induced transformation is then represented by

$$\boxed{\mathbf{X}^c = \alpha^i(\mathbf{q})\frac{\partial}{\partial q^i} + \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\alpha^i(\mathbf{q})\right)\frac{\partial}{\partial \dot{q}^j}}$$

'complete lift of X'

Noether symmetry approach and spherical symmetry

A function $f(\mathbf{q}, \dot{\mathbf{q}})$ is invariant under the transformation X^c if

$$L_{\mathbf{X}^c} f \overset{\text{def}}{=} \alpha^i(\mathbf{q}) \frac{\partial f}{\partial q^i} + \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} \alpha^i(\mathbf{q}) \right) \frac{\partial f}{\partial \dot{q}^j} = 0,$$

In particular, if $L_{\mathbf{X}^c} \mathcal{L} = 0$, X^C is said to be a symmetry for the dynamics derived by Lagrangian.

In order to see how Noether's theorem and cyclic variables are related, let us consider a Lagrangian $\mathcal{L}$ and its Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial \mathcal{L}}{\partial \dot{q}^j} - \frac{\partial \mathcal{L}}{\partial q^j} = 0.$$

Let us also consider the vector field.
Contracting with α^i 's gives

$$\alpha^j \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial \mathcal{L}}{\partial \dot{q}^j} - \frac{\partial \mathcal{L}}{\partial q^j} \right) = 0.$$

with

$$\alpha^j \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial \mathcal{L}}{\partial \dot{q}^j} = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\alpha^j \frac{\partial \mathcal{L}}{\partial \dot{q}^j} \right) - \left(\frac{\mathrm{d}\alpha^j}{\mathrm{d}\lambda} \right) \frac{\partial \mathcal{L}}{\partial \dot{q}^j},$$

The immediate consequence is the Noether theorem: if $L_{\mathbf{X}} \mathcal{L} = 0$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = L_{\mathbf{X}} \mathcal{L}. \quad \Rightarrow \quad \Sigma_0 = \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$$

constant of motion

Noether symmetry approach and spherical symmetry

In our case we have that $q = (A, B, M, R)$ and $q' = (A', B', M', R')$ are respectively the generalized positions and velocities associated with $\mathcal{L}$.

If we assume the spherical symmetry, the role of the affine parameter is played by the coordinate radius r.

The configuration space is given by $\mathcal{Q} = \{A, M, R\}$ and the tangent space by $T\mathcal{Q} = \{A, A', M, M', R, R'\}$.

if we choose $\boxed{f(R) = f_0 R^s,}$

There exists, a Noether symmetry and a related constant of motion for $s = 1$ $\boxed{\Sigma_0 = \frac{2GM}{c^2}}$

In the Einstein gravity, the Noether symmetry gives, as a conserved quantity, the Schwarzschild radius or the mass of the gravitating system.

This result can be assumed as a consistency check!!!

Noether symmetry approach and spherical symmetry

We can find out general solutions for the field equations giving the dependence of the Ricci scalar on the radial coordinate r

a solution is found for $\boxed{s = 5/4,}$ $\boxed{M = r^2,}$ $\boxed{R = 5r^{-2},}$

obtaining the spherically symmetric spacetime

$$\boxed{ds^2 = (\alpha + \beta r)\,dt^2 - \frac{1}{2}\frac{\beta r}{\alpha + \beta r}\,dr^2 - r^2\,d\Omega}$$

The same procedure can be worked out when Noether symmetries are identified.

Axial symmetry derived from spherical symmetry

It is possible to obtain an axially symmetric solution starting from spherical symmetry adopting the method developed by Newman and Janis in GR.

Newman ET and Janis AI 1965 J.Math.Phys.6915

Such an algorithm can be applied to a static spherically symmetric metric considered as a 'seed' metric.

$$\mathrm{d}s^2 = \mathrm{e}^{2\phi(r)}\,\mathrm{d}t^2 - \mathrm{e}^{2\lambda(r)}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega,$$

Can be written in the so-called Eddington-Finkelstein coordinates (u, r, θ, ϕ), i.e. the g_{rr} component is eliminated by a change of coordinates and a cross term is introduced.

$$\mathrm{d}t = \mathrm{d}u + F(r)\,\mathrm{d}r \qquad F(r) = \pm\mathrm{e}^{\lambda(r)-\phi(r)}$$

Axial symmetry derived from spherically symmetric solutions

Metric becomes

$$\boxed{\mathrm{d}s^2 = \mathrm{e}^{2\phi(r)}\,\mathrm{d}u^2 \pm 2\,\mathrm{e}^{\lambda(r)+\phi(r)}\,\mathrm{d}u\,\mathrm{d}r - r^2\,\mathrm{d}\Omega.}$$

The metric tensor for the line element in null coordinates is

$$\boxed{g^{\mu\nu} = \begin{pmatrix} 0 & \pm\mathrm{e}^{-\lambda(r)-\phi(r)} & 0 & 0 \\ \pm\mathrm{e}^{-\lambda(r)-\phi(r)} & -\mathrm{e}^{-2\lambda(r)} & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/(r^2\sin^2\theta) \end{pmatrix}}$$

Matrix can be written in terms of a null tetrad as

$$\boxed{g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu,}$$

l^μ, n^μ, m^μ and $\bar{m}^\mu$ are the vectors satisfying the conditions

$$\boxed{l_\mu l^\mu = m_\mu m^\mu = n_\mu n^\mu = 0, \qquad l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1, \qquad l_\mu m^\mu = n_\mu m^\mu = 0.}$$

Axial symmetry derived from spherically symmetric solutions

At any point in space, the tetrad can be chosen in the following manner:

- l^μ is the outward null vector tangent to the cone,
- n^μ is the inward null vector pointing toward the origin, and
- m^μ and $\bar{m}^\mu$ are the vectors tangent to the two-dimensional sphere defined by the constants r and u.

For the our spacetime, the tetrad null vectors can be

$$\begin{cases} l^\mu = \delta_1^\mu \\ n^\mu = -\frac{1}{2}\,\mathrm{e}^{-2\lambda(r)}\delta_1^\mu + \mathrm{e}^{-\lambda(r)-\phi(r)}\delta_0^\mu \\ m^\mu = \frac{1}{\sqrt{2}r}\left(\delta_2^\mu + \frac{\mathrm{i}}{\sin\theta}\delta_3^\mu\right) \\ \bar{m}^\mu = \frac{1}{\sqrt{2}r}\left(\delta_2^\mu - \frac{\mathrm{i}}{\sin\theta}\delta_3^\mu\right). \end{cases}$$

Axial symmetry derived from spherically symmetric solutions

Now we need to extend the set of coordinates $x_\mu=(u, r, \theta, \phi)$ replacing the real radial coordinate by a complex variable.

Then the tetrad null vectors become

$$\begin{cases} l^\mu = \delta_1^\mu \\ n^\mu = -\frac{1}{2}\,\mathrm{e}^{-2\lambda(r,\bar{r})}\delta_1^\mu + \mathrm{e}^{-\lambda(r,\bar{r})-\phi(r,\bar{r})}\delta_0^\mu \\ m^\mu = \frac{1}{\sqrt{2}\bar{r}}\left(\delta_2^\mu + \frac{\mathrm{i}}{\sin\theta}\delta_3^\mu\right) \\ \bar{m}^\mu = \frac{1}{\sqrt{2}r}\left(\delta_2^\mu - \frac{\mathrm{i}}{\sin\theta}\delta_3^\mu\right). \end{cases}$$

A new metric is obtained by making a complex coordinate transformation

$$x^\mu \to \tilde{x}^\mu = x^\mu + \mathrm{i}y^\mu(x^\sigma)$$

and simultaneously let the null tetrad vectors $Z_a^\mu = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$

with a = 1, 2, 3, 4, undergo the transformation

$$Z_a^\mu \to \tilde{Z}_a^\mu(\tilde{x}^\sigma, \bar{\tilde{x}}^\sigma) = Z_a^\rho \frac{\partial \tilde{x}^\mu}{\partial x^\rho}.$$

Axial symmetry derived from spherically symmetric solutions

The effect of the 'tilde transformation' is the generation of a new metric whose components are the (real) functions of complex variables, that is,

$$\boxed{g_{\mu\nu} \to \tilde{g}_{\mu\nu} : \tilde{\mathbf{x}} \times \tilde{\mathbf{x}} \mapsto \mathbb{R}}$$

with

$$\boxed{\tilde{Z}^{\mu}_{a}(\tilde{x}^{\sigma}, \bar{\tilde{x}}^{\sigma})\big|_{\mathbf{x}=\tilde{\mathbf{x}}} = Z^{\mu}_{a}(x^{\sigma}).}$$

For our aims, we make the choice

$$\tilde{x}^{\mu} = x^{\mu} + \mathrm{i}a\left(\delta^{\mu}_{1} - \delta^{\mu}_{0}\right)\cos\theta \to \begin{cases} \tilde{u} = u + \mathrm{i}a\cos\theta \\ \tilde{r} = r - \mathrm{i}a\cos\theta \\ \tilde{\theta} = \theta \\ \tilde{\phi} = \phi \end{cases}$$

if we choose $\boxed{\tilde{r} = \bar{\tilde{r}}}$

the tetrad null vectors become

$$\begin{cases} \tilde{l}^{\mu} = \delta^{\mu}_{1} \\ \tilde{n}^{\mu} = -\frac{1}{2}\,\mathrm{e}^{-2\lambda(\tilde{r},\theta)}\delta^{\mu}_{1} + \mathrm{e}^{-\lambda(\tilde{r},\theta)-\phi(\tilde{r},\theta)}\delta^{\mu}_{0} \\ \tilde{m}^{\mu} = \frac{1}{\sqrt{2}(\tilde{r}-\mathrm{i}a\cos\theta)}\left[\mathrm{i}a\left(\delta^{\mu}_{0} - \delta^{\mu}_{1}\right)\sin\theta + \delta^{\mu}_{2} + \frac{\mathrm{i}}{\sin\theta}\delta^{\mu}_{3}\right] \\ \bar{\tilde{m}}^{\mu} = \frac{1}{\sqrt{2}(\tilde{r}+\mathrm{i}a\cos\theta)}\left[-\mathrm{i}a\left(\delta^{\mu}_{0} - \delta^{\mu}_{1}\right)\sin\theta + \delta^{\mu}_{2} - \frac{\mathrm{i}}{\sin\theta}\delta^{\mu}_{3}\right]. \end{cases}$$

Axial symmetry derived from spherically symmetric solutions

The new metric, with the coordinates $\tilde{x}^\mu = (\tilde{u}, \tilde{r}, \theta, \phi)$., is

$$\tilde{g}^{\mu\nu} = \begin{pmatrix} -\frac{a^2\sin^2\theta}{\Sigma^2} & e^{-\lambda(\tilde{r},\theta)-\phi(\tilde{r},\theta)} + \frac{a^2\sin^2\theta}{\Sigma^2} & 0 & -\frac{a}{\Sigma^2} \\ \cdot & -e^{-2\lambda(\tilde{r},\theta)} - \frac{a^2\sin^2\theta}{\Sigma^2} & 0 & \frac{a}{\Sigma^2} \\ \cdot & \cdot & -\frac{1}{\Sigma^2} & 0 \\ \cdot & \cdot & \cdot & -\frac{1}{\Sigma^2\sin^2\theta} \end{pmatrix}$$

where $\Sigma = \sqrt{\tilde{r}^2 + a^2\cos^2\theta}$.

In the covariant form,

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} e^{2\phi(\tilde{r},\theta)} & e^{\lambda(\tilde{r},\theta)+\phi(\tilde{r},\theta)} & 0 & a\, e^{\phi(\tilde{r},\theta)}[e^{\lambda(\tilde{r},\theta)} - e^{\phi(\tilde{r},\theta)}]\sin^2\theta \\ \cdot & 0 & 0 & -a\, e^{\phi(\tilde{r},\theta)+\lambda(\tilde{r},\theta)}\sin^2\theta \\ \cdot & \cdot & -\Sigma^2 & 0 \\ \cdot & \cdot & \cdot & -[\Sigma^2 + a^2\sin^2\theta\, e^{\phi(\tilde{r},\theta)}(2\,e^{\lambda(\tilde{r},\theta)} - e^{\phi(\tilde{r},\theta)})]\sin^2\theta \end{pmatrix}$$

The form of this metric gives the general result of the Newman-Janis algorithm starting from any spherically symmetric 'seed' metric.

Axially symmetric solutions in f(R)-gravity

Now our task is to show that such an approach can be used to derive axially symmetric solutions also in $f(R)$-gravity.

$$g^{\mu\nu} = \begin{pmatrix} 0 & \sqrt{\frac{2}{\beta r}} & 0 & 0 \\ . & -2-\frac{2\alpha}{\beta r} & 0 & 0 \\ . & . & -1/r^2 & 0 \\ . & . & . & -1/(r^2 \sin^2\theta) \end{pmatrix}$$

The complex tetrad null vectors are now

$$\begin{cases} l^\mu = \delta_1^\mu \\ n^\mu = -\left[1+\frac{\alpha}{\beta}\left(\frac{1}{\bar{r}}+\frac{1}{r}\right)\right]\delta_1^\mu + \sqrt{\frac{2}{\beta}}\frac{1}{\sqrt[4]{\bar{r}r}}\delta_0^\mu \\ m^\mu = \frac{1}{\sqrt{2\bar{r}}}\left(\delta_2^\mu + \frac{\mathrm{i}}{\sin\theta}\delta_3^\mu\right). \end{cases}$$

By computing the complex coordinate transformation , the tetrad null vectors become

$$\begin{cases} \tilde{l}^\mu = \delta_1^\mu \\ \tilde{n}^\mu = -\left[1+\frac{\alpha}{\beta}\frac{\mathrm{Re}\{\tilde{r}\}}{\Sigma^2}\right]\delta_1^\mu + \sqrt{\frac{2}{\beta}}\frac{1}{\sqrt{\Sigma}}\delta_0^\mu \\ \tilde{m}^\mu = \frac{1}{\sqrt{2}(\tilde{r}+\mathrm{i}a\cos\theta)}\left[\mathrm{i}a\left(\delta_0^\mu - \delta_1^\mu\right)\sin\theta + \delta_2^\mu + \frac{\mathrm{i}}{\sin\theta}\delta_3^\mu\right]. \end{cases}$$

Axially symmetric solutions in f(R)-gravity

Now by performing the same procedure as in the previous section, we derive an axially symmetric metric but starting from the spherically symmetric metric, that is,

$$g_{\mu\nu} = \begin{pmatrix} \frac{r(\alpha+\beta r)+a^2\beta\cos^2\theta}{\Sigma} & 0 & 0 & \frac{a(-2\alpha r-2\beta\Sigma^2+\sqrt{2\beta}\Sigma^{3/2})\sin^2\theta}{2\Sigma} \\ \cdot & -\frac{\beta\Sigma^2}{2\alpha r+\beta(a^2+r^2+\Sigma^2)} & 0 & 0 \\ \cdot & \cdot & -\Sigma^2 & 0 \\ \cdot & \cdot & \cdot & -\left[\Sigma^2-\frac{a^2(\alpha r+\beta\Sigma^2-\sqrt{2\beta}\Sigma^{3/2})\sin^2\theta}{\Sigma}\right]\sin^2\theta \end{pmatrix}.$$

This is nothing else but an example; the method is general and can be extended to any spherically symmetric solution derived in f (R)-gravity.

Physical applications

We take into account a freely falling particle moving in the spacetime described by metric

We make explicit use of the Hamiltonian formalism

Given a metric $g_{\mu\nu}$, the motion along the geodesics can be described by the Lagrangian

$$\boxed{\mathcal{L}(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$$

canonical momenta and the Hamiltonian function

$$\boxed{p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g^{\mu\nu} p_\mu p_\nu} \qquad \boxed{\mathcal{H} = p_\mu \dot{x}^\mu - \mathcal{L}} \quad \Longrightarrow \quad \boxed{\mathcal{H} = \frac{1}{2} p_\mu p_\nu g^{\mu\nu}}$$

The Hamiltonian results explicitly independent of time and it is

$$\boxed{\mathcal{H} = -\frac{1}{2} m^2}$$

the rest mass m is a constant

Physical applications

The geodesic equations are

$$\frac{dx^\mu}{d\lambda} = \frac{\partial \mathcal{H}}{\partial p_\mu} = g^{\mu\nu} p_\nu = p^\mu,$$

$$\frac{dp_\mu}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial x\mu} = -\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^\mu} p_\alpha p_\beta = g^{\gamma\beta}\Gamma^\alpha_{\mu\gamma} p_\alpha p_\beta.$$

Using the above definitions,

$$H = -p_0 = \left[\frac{p_i g^{0i}}{g^{00}} + \left[\left(\frac{p_i g^{0i}}{g^{00}} \right)^2 - \frac{m^2 + p_i p_j g^{ij}}{g^{00}} \right] \right]^{1/2}$$

with the equations of motion

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}$$

that give the orbits

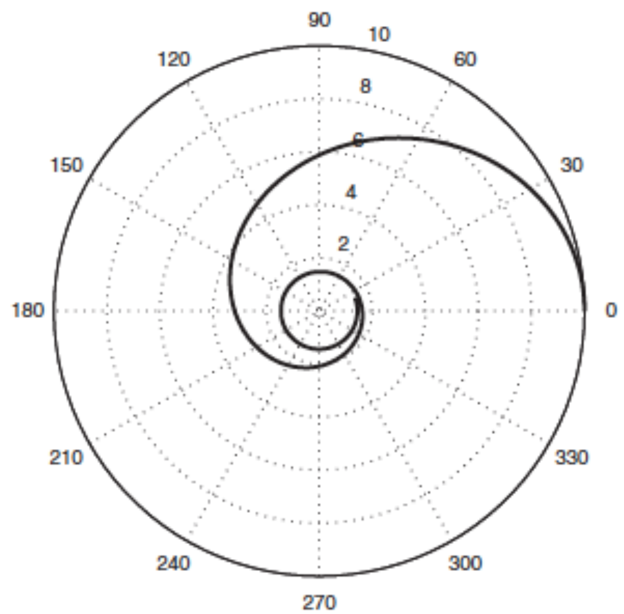

Figure 1. Relative motion of the test particle with $m = 1$.

Conclusions

- Very few exact solutions exsist in Extended Theories of Gravity in particular in f(R) gravity.
- The Newman–Janis method can be used to derive axially symmetric solutions in GR and in f (R)-gravity.
- The method does not depend on the field equations but directly works on the solutions that, a posteriori, have to be checked to fulfill the field equations.
- The key point of the method is to find out a suitable complex transformation of coordinates which corresponds to the reduction of number of independent Killing vectors.
- Other generating techniques in order to get solutions in f (R)-gravity.
- Physical proprerties have to be fully explored.