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*Alternative point of view about the classical theory
of Gravitation: The Fourth Order Gravity*

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Fourth Order Gravity (FOG)

From action principle $A^f = \int d^4x \sqrt{-g} \left[f(X, Y, Z) + \chi \mathcal{L}_m \right]$

where $X = R$ (Ricci scalar), $Y = R_{\alpha\beta} R^{\alpha\beta}$ (Ricci tensor square) and $Z = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ (Riemann square)

The field equations are

$$H_{\mu\nu} = f_X R_{\mu\nu} - \frac{f}{2} g_{\mu\nu} - f_{X;\mu\nu} + g_{\mu\nu} \square f_X + 2f_Y R_{\mu}^{\alpha} R_{\alpha\nu} - 2[f_Y R^{\alpha}{}_{(\mu};{}_{\nu)\alpha}] + \square[f_Y R_{\mu\nu}] + [f_Y R_{\alpha\beta}]{}^{;\alpha\beta} g_{\mu\nu} \\ + 2f_Z R_{\mu\alpha\beta\gamma} R_{\nu}^{\alpha\beta\gamma} - 4[f_Z R_{\mu}^{\alpha\beta}{}_{\nu}]{}_{;\alpha\beta} = \chi T_{\mu\nu}$$

and the trace is given by

$$H = g^{\alpha\beta} H_{\alpha\beta} = f_X R + 2f_Y R_{\alpha\beta} R^{\alpha\beta} + 2f_Z R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 2f + \square[3f_X + f_Y R] + 2[(f_Y + 2f_Z) R^{\alpha\beta}]{}_{;\alpha\beta} = \chi T$$

Why we have a theory of fourth order?

In GR theory we have naturally four divergences ... while in FOG we must make them, but ...



Spherically symmetric solutions

We are interested to investigate solutions in the Newtonian limit. The most general symmetric solution can be written as follows

$$ds^2 = g_1(t, |\mathbf{x}|) dt^2 + g_2(t, |\mathbf{x}|) dt \mathbf{x} \cdot d\mathbf{x} + g_3(t, |\mathbf{x}|) (\mathbf{x} \cdot d\mathbf{x})^2 + g_4(t, |\mathbf{x}|) d|\mathbf{x}|^2$$

but we have an arbitrariness in the choice of coordinates. Then we can use also the expressions of metric (isotropic coordinates):

$$ds^2 = g_{tt}(t', r') dt'^2 - g_{ij}(t', r') dx^i dx^j$$

and (standard coordinates):

$$ds^2 = g_{tt}(t', r'') dt'^2 - g_{rr}(t', r'') dr''^2 - r''^2 d\Omega$$



Spherical symmetry in $f(R)$ -gravity

If the metric is time-independent, the definition of Ricci scalar gives rise to a differential equation (Bernoulli equation):

$$b'(r) + \left\{ \frac{r^2 a'(r)^2 - 4a(r)^2 - 2ra(r)[2a(r)' + ra(r)']}{ra(r)[4a(r) + ra'(r)]} \right\} b(r) + \left\{ \frac{2a(r)}{r} \left[\frac{2 + r^2 R(r)}{4a(r) + ra'(r)} \right] \right\} b(r)^2 \doteq b'(r) + h(r)b(r) + l(r)b(r)^2 = 0$$

This is a relation linking the metric potentials and the Ricci scalar.

If we consider theories without cosmological constant + Hilbert-Einstein term and if $\lim_{R \rightarrow 0} f \sim R^2$

We have a class of solutions $b(r) = \frac{\exp[-\int dr h(r)]}{K + 4 \int \frac{dr a(r) \exp[-\int dr h(r)]}{r[a(r)+ra'(r)]}}$

In this case, the GR limit is not present!

Spherical symmetry of $f(R)$ -gravity with constant Ricci scalar



Whenever the gravitational potential $g_{tt}(t, r)$ is described by a separable functions and $g_{rr}(t, r)$ is time - independent, by the definition of the Ricci scalar, one gets that $R = \text{constant}$ and, at the same time, the final solutions of the field equations will be static if the spherical symmetry is invoked: **Birkhoff theorem** is valid also for this particular class of f - gravity theories.

$$\begin{cases} R_{\mu\nu} + \lambda g_{\mu\nu} = q \mathcal{X} T_{\mu\nu} \\ R_0 = q \mathcal{X} T - 4\lambda \end{cases}$$

$$ds^2 = \left(1 + \frac{k_1}{r} + \frac{q\mathcal{X}\rho - \lambda}{3} r^2 \right) dt^2 - \frac{dr^2}{1 + \frac{k_1}{r} + \frac{q\mathcal{X}\rho - \lambda}{3} r^2} - r^2 d\Omega$$

In other words, any f - theory, in the case of constant curvature scalar, exhibits solutions with cosmological constant as the Schwarzschild - de Sitter solution.

Spherical symmetry of f -gravity with constant Ricci scalar

In standard GR- gravity, the constant curvature solutions different from zero are obtained only in presence of matter because of the proportionality of the Ricci scalar to the trace of the energy-momentum tensor of matter, or, on the other side, one can get a similar situation in presence of the cosmological constant. Actually the big difference between GR and higher order gravity is that, **the Schwarzschild-de Sitter solution is not necessarily given by the cosmological term** while the effect of an “effective” cosmological constant, in the low energy limit, can be played by the higher order derivative contributions evaluated on Ricci constant backgrounds.



A perturbative approach to f -gravity

If we suppose, for the metric, an expression $g_{\mu\nu} = g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(1)}$ and compute the perturbations in the field equations, we obtain the equations:

$$f'^{(0)} R_{\mu\nu}^{(0)} - \frac{1}{2} g_{\mu\nu}^{(0)} f^{(0)} + \mathcal{H}_{\mu\nu}^{(0)} = \chi T_{\mu\nu}^{(0)}$$

$$f'^{(0)} \left\{ R_{\mu\nu}^{(1)} - \frac{1}{2} g_{\mu\nu}^{(0)} R^{(1)} \right\} + f''^{(0)} R^{(1)} R_{\mu\nu}^{(0)} - \frac{1}{2} f^{(0)} g_{\mu\nu}^{(1)} + \mathcal{H}_{\mu\nu}^{(1)} = \chi T_{\mu\nu}^{(1)}$$

Besides if we consider $f = R + \mathcal{F}(R)$ The set of equations is

$$R_{\mu\nu}^{(0)} - \frac{1}{2} R^{(0)} g_{\mu\nu}^{(0)} = G_{\mu\nu}^{(0)} = \chi T_{\mu\nu}^{(0)} \quad (\text{GR equations})$$

$$R_{\mu\nu}^{(1)} - \frac{1}{2} g_{\mu\nu}^{(0)} R^{(1)} - \frac{1}{2} g_{\mu\nu}^{(1)} R^{(0)} - \frac{1}{2} g_{\mu\nu}^{(0)} \mathcal{F}^{(0)} + \mathcal{F}'^{(0)} R_{\mu\nu}^{(0)} + \mathcal{H}_{\mu\nu}^{(1)} = \chi T_{\mu\nu}^{(1)}$$

What is wrong in PPN-parameterization of Higher Order Gravity?



The Eddington parameters are defined by relations

$$ds^2 \simeq \left[1 - \alpha \frac{r_g}{r'} + \frac{\beta}{2} \left(\frac{r_g}{r'} \right)^2 + \dots \right] dt^2 - \left[1 + \gamma \frac{r_g}{r'} + \dots \right] \left[dr'^2 + r'^2 d\Omega \right]$$

The Eddington idea born when Lagrangians proportional to Ricci scalar were investigated.

TODAY the parameterization by Eddington parameters could be misinterpreted.



$f(R)$ -gravity and O'Hanlon theory

The f -gravity is analogous O' Hanlon theory and not to Brans-Dicke theory.

$$\left\{ \begin{array}{l} \mathcal{A}_{JF}^f = \int d^4x \sqrt{-g} \left[f + \mathcal{X} \mathcal{L}_m \right] \\ \mathcal{A}_{JF}^{BD} = \int d^4x \sqrt{-g} \left[\phi R - \omega_{BD} \frac{\phi_{;\alpha} \phi^{;\alpha}}{\phi} + \mathcal{X} \mathcal{L}_m \right] \\ \mathcal{A}_{JF}^{OH} = \int d^4x \sqrt{-g} \left[\phi R + V(\phi) + \mathcal{X} \mathcal{L}_m \right] \end{array} \right. \quad \left\{ \begin{array}{l} \phi = f' \\ V(\phi) = f - f' R \\ \phi \frac{dV(\phi)}{d\phi} - 2V(\phi) = f' R - 2f \end{array} \right.$$

If we perform the Newtonian limit of O' Hanlon theory and come back to f -theory approach the solution

$$\left\{ \begin{array}{l} g_{tt} = 1 - \frac{2}{3a} \frac{r_g}{|\mathbf{x}|} - \sqrt{\frac{\pi}{2}} \frac{1}{3a} \frac{r_g e^{-\lambda|\mathbf{x}|}}{|\mathbf{x}|} \\ g_{ij} = - \left\{ 1 + \frac{1}{3a} \frac{r_g}{|\mathbf{x}|} - \sqrt{\frac{\pi}{2}} \frac{r_g}{3a} \left[\left(\frac{1}{|\mathbf{x}|} - \frac{2}{\lambda^2 |\mathbf{x}|^3} - \frac{2}{\lambda |\mathbf{x}|^2} \right) e^{-\lambda|\mathbf{x}|} - \frac{2}{\lambda^2 |\mathbf{x}|^3} \right] \right\} \delta_{ij} \\ + \frac{(2\pi)^{1/2} r_g}{3a} \left[\left(\frac{1}{|\mathbf{x}|} + \frac{3}{\lambda |\mathbf{x}|^2} + \frac{3}{\lambda^2 |\mathbf{x}|^3} \right) e^{-\lambda|\mathbf{x}|} - \frac{3}{\lambda^2 |\mathbf{x}|^3} \right] \frac{x_i x_j}{|\mathbf{x}|^2} \end{array} \right.$$

If we turn off the modification in the theory we don't find the GR. The same situation for BD theory! *So the same problem is present also in the conformal transformations.*



The Noether Symmetry approach to f -gravity

We worked out an approach to obtain time-independent spherically symmetric solutions in f -gravity. In order to develop such an approach, we need to deduce a point-like Lagrangian from the general action. Such a Lagrangian can be obtained by imposing the spherical symmetry in the field action. As a consequence, the infinite number of degrees of freedom of the original field theory will be reduced to a finite number.

$$\mathcal{L} = -\frac{A^{1/2} f_R}{2MB^{1/2}} M'^2 - \frac{f_R}{A^{1/2} B^{1/2}} A' M' - \frac{M f_{RR}}{A^{1/2} B^{1/2}} A' R' + \\ -\frac{2A^{1/2} f_{RR}}{B^{1/2}} R' M' - A^{1/2} B^{1/2} [(2 + MR) f_R - Mf]$$

The Euler-Lagrange equations are in terms of the function A , B , M , R . The field equation for R corresponds to the constraint among the configuration coordinates.

It is worth noting that the Hessian determinant is zero! The point-like Lagrangian does not depend on B . In other words, B does not contribute to dynamics and its equation has to be considered as a further constraint equation.



The Noether Symmetry approach to f -gravity

The field-equations approach and the point-like Lagrangian approach differ since the symmetry, in our case the spherical one, can be imposed whether in the field equations, after standard variation with respect to the metric, or directly into the Lagrangian, which becomes point-like.

Field equations approach

$$\delta \int d^4x \sqrt{-g} f = 0$$

$$H_{\mu\nu} = \partial_\rho \frac{\partial(\sqrt{-g}f)}{\partial_\rho g^{\mu\nu}} - \frac{\partial(\sqrt{-g}f)}{\partial g^{\mu\nu}} = 0$$

$$H = g^{\alpha\beta} H_{\alpha\beta} = 0$$

$$H_{tt} = 0$$

$$H_{rr} = 0$$

$$H_{\theta\theta} = \csc^2 \theta H_{\phi\phi} = 0$$

$$H = A^{-1} H_{tt} - B^{-1} H_{rr} - 2M^{-1} \csc^2 \theta H_{\phi\phi} = 0$$

Point-like Lagrangian approach

$$\delta \int dr \mathcal{L} = 0$$

$$\frac{d}{dr} \nabla_{q'} \mathcal{L} - \nabla_{q'} \mathcal{L} = 0$$

$$E_{\mathcal{L}} = \underline{q}' \cdot \nabla_{q'} \mathcal{L} - \mathcal{L}$$

$$\frac{d}{dr} \frac{\partial \mathcal{L}}{\partial A'} - \frac{\partial \mathcal{L}}{\partial A} = 0$$

$$\frac{d}{dr} \frac{\partial \mathcal{L}}{\partial B'} - \frac{\partial \mathcal{L}}{\partial B} \propto E_{\mathcal{L}} = 0$$

$$\frac{d}{dr} \frac{\partial \mathcal{L}}{\partial M'} - \frac{\partial \mathcal{L}}{\partial M} = 0$$

A combination of the above equations



The Noether Symmetry approach to f -gravity

We find a relation between the metric potentials

$$B = \frac{2M^2 f_{RR} A' R' + 2M f_R A' M' + 4AM f_{RR} M' R' + A f_R M'^2}{2AM[(2 + MR)f_R - Mf]}$$

If $f = R$ we find the usual condition for A and B . Now it is possible to reduce the configuration space of dynamics and we have

$$\mathbf{L} = \underline{q}'^t \hat{\mathbf{L}} \underline{q}' = \frac{[(2 + MR)f_R - fM]}{M} \\ \times [2M^2 f_{RR} A' R' + 2MM'(f_R A' + 2A f_{RR} R') + A f_R M'^2]$$

By applying the Noether Symmetry Approach we find a constant motion for a power law f

$$\Sigma_0 = \underline{\alpha} \cdot \nabla_{\underline{q}'} \mathbf{L} = \\ = 2skMR^{2s-3} [2s + (s-1)MR] [(s-2)RA' - (2s^2 - 3s + 1)AR']$$

The solution, for $s = 5/2$, is

$$ds^2 = \frac{1}{\sqrt{5}}(k_2 + k_1 r) dt^2 - \frac{1}{2} \left(\frac{1}{1 + \frac{k_2}{k_1 r}} \right) dr^2 - r^2 d\Omega$$

Spherically symmetric solution with space-dependent scalar curvature for f -gravity by perturbations

If the Ricci scalar is only space-dependent we find as solution

$$a(r) = \frac{b(r) e^{-\frac{2}{3} \int \frac{[R+(2\mathcal{F}-R\mathcal{F}')]b(r)}{R'\mathcal{F}''} dr}}{r^4 R'^2 \mathcal{F}''^2}$$

$$f = R + \mathcal{F}(R), \text{ with } \mathcal{F}(R) \ll R$$

$$b(r) = -\frac{3(rR'\mathcal{F}'')_{,r}}{rR}$$

Also in this case, the Birkhoff theorem is satisfied.

The only off-diagonal component equation non identically vanishing is

$$\frac{d}{dr} \left(r^2 f' \right) \dot{g}_{rr}(t, r) = 0$$

Other choices are not possible: we would find incompatibilities.



Weak field limit of FOG

There are two weak field limits:

- Newtonian and Post Newtonian limit
- Post Minkowskian limit → Gravitational waves

If one takes into account a system of gravitationally interacting massive particles, the kinetic energy will be, roughly, of the same order of magnitude as the typical potential energy. As a consequence

$$\bar{v}^2 \sim \frac{GM}{\bar{r}}$$

The Newtonian and the Post-Newtonian approximation can be described as a method for obtaining the motion of the system to higher approximations than the first order⁴ with respect to the quantities velocity square and / or gravitational potential assumed small with respect to the squared light speed.

This approximation is sometimes referred to as an expansion in inverse powers of the light speed.



Newtonian and Post-Newtonian Limit of FOG

First consequences are $\frac{\partial}{\partial t} \sim \mathbf{v} \cdot \nabla$ and $\frac{|\partial/\partial t|}{|\nabla|} \sim \mathcal{O}(1)$

The geodesic equation $\frac{d^2 x^\mu}{ds^2} + \Gamma_{\sigma\tau}^\mu \frac{dx^\sigma}{ds} \frac{dx^\tau}{ds} = 0$ can be rewritten

$$\frac{d^2 x^i}{dt^2} = -\Gamma_{tt}^i - 2\Gamma_{tm}^i \frac{dx^m}{dt} - \Gamma_{mn}^i \frac{dx^m}{dt} \frac{dx^n}{dt} + \left[\Gamma_{tt}^t + 2\Gamma_{tm}^t \frac{dx^m}{dt} + 2\Gamma_{mn}^t \frac{dx^m}{dt} \frac{dx^n}{dt} \right] \frac{dx^i}{dt}$$

Then at lowest level we have the Newtonian Mechanics $\frac{d^2 x^i}{dt^2} \simeq -\Gamma_{tt}^i \simeq -\frac{1}{2} \frac{\partial g_{tt}}{\partial x^i}$

The metric tensor becomes

$$\begin{cases} g_{tt}(t, \mathbf{x}) \simeq 1 + g_{tt}^{(2)}(t, \mathbf{x}) + g_{tt}^{(4)}(t, \mathbf{x}) + \mathcal{O}(6) \\ g_{ti}(t, \mathbf{x}) \simeq g_{ti}^{(3)}(t, \mathbf{x}) + \mathcal{O}(5) \\ g_{ij}(t, \mathbf{x}) \simeq -\delta_{ij} + g_{ij}^{(2)}(t, \mathbf{x}) + \mathcal{O}(4) \end{cases}$$



Newtonian of $f(R)$ -Gravity

We start with a metric as follows

Any $f(R)$ can be developed as Taylor series in some point

$$f = \sum_n \frac{f^n(R_0)}{n!} (R - R_0)^n \simeq f_0 + f_1 R + f_2 R^2 + f_3 R^3 + \dots$$

$$\left\{ \begin{array}{l} g_{tt}(t, r) \simeq 1 + g_{tt}^{(2)}(t, r) + g_{tt}^{(4)}(t, r) \\ g_{rr}(t, r) \simeq -1 + g_{rr}^{(2)}(t, r) \\ g_{\theta\theta}(t, r) = -r^2 \\ a_{\phi\phi}(t, r) = -r^2 \sin^2 \theta \end{array} \right.$$

$$\left\{ \begin{array}{l} f_1 r R^{(2)} - 2f_1 g_{tt,r}^{(2)} + 8f_2 R_{,r}^{(2)} - f_1 r g_{tt,rr}^{(2)} + 4f_2 r R^{(2)} = 0 \\ f_1 r R^{(2)} - 2f_1 g_{rr,r}^{(2)} + 8f_2 R_{,r}^{(2)} - f_1 r g_{tt,rr}^{(2)} = 0 \\ 2f_1 g_{rr}^{(2)} - r[f_1 r R^{(2)} - f_1 g_{tt,r}^{(2)} - f_1 g_{rr,r}^{(2)} + 4f_2 R_{,r}^{(2)} + 4f_2 r R_{,rr}^{(2)}] = 0 \\ f_1 r R^{(2)} + 6f_2 [2R_{,r}^{(2)} + r R_{,rr}^{(2)}] = 0 \\ 2g_{rr}^{(2)} + r[2g_{tt,r}^{(2)} - r R^{(2)} + 2g_{rr,r}^{(2)} + r g_{tt,rr}^{(2)}] = 0 \end{array} \right.$$

If the Lagrangian is expanded around a vanishing value of the Ricci scalar ($R_0 = 0$), the field equation at lowest order implies that the cosmological constant contribution has to be zero whatever is the $f(R)$ -Gravity model

$$f_0 = 0$$



Newtonian of $f(R)$ -Gravity

The solution of metric tensor is

$$\begin{cases} ds^2 = \left[1 - \frac{r_g}{f_1 r} + \frac{\delta_2(t)}{3m} \frac{e^{-mr}}{mr} \right] dt^2 - \left[1 + \frac{r_g}{f_1 r} + \frac{\delta_2(t)}{3m} \frac{mr+1}{mr} e^{-mr} \right] dr^2 - r^2 d\Omega \\ R = \frac{\delta_2(t) e^{-mr}}{r} \end{cases} \quad m \doteq \sqrt{-\frac{f_1}{6f_2}}$$

At third order the field equation states the relation between the “rr” component and the Ricci scalar $f_1 g_{rr,t}^{(2)} + 2f_2 r R_{,tr}^{(2)} = 0$

In fact, generally, if the “rr” component is time-independent and the “tt” component is the product of two functions (one of time and other one of the space) the Ricci scalar is time independent. Indeed exists every a redefinition of time gives back a metric time-independent. **Is not verified the Birkhoff theorem in $f(R)$?** It works only at Newtonian level.

Therefore, the Birkhoff theorem is not a general result for higher order gravity but, on the other hand, in the limit of small velocities and weak fields (which is enough to deal with the Solar System gravitational experiments), one can assume that the gravitational potential is effectively time-independent.

Newtonian and Post-Newtonian of $f(R)$ -Gravity in isotropic coordinates



The metric tensor is developed as

$$g_{\mu\nu} \sim \begin{pmatrix} 1 + g_{tt}^{(2)}(t, \mathbf{x}) + g_{tt}^{(4)}(t, \mathbf{x}) + \dots & g_{ti}^{(3)}(t, \mathbf{x}) + \dots \\ g_{ti}^{(3)}(t, \mathbf{x}) + \dots & -\delta_{ij} + g_{ij}^{(2)}(t, \mathbf{x}) + \dots \end{pmatrix}$$

The field equations at
“Newtonian level” are

$$\left\{ \begin{array}{l} H_{tt}^{(2)} = f'(0)R_{tt}^{(2)} - \frac{f'(0)}{2}R^{(2)} - f''(0)\Delta R^{(2)} = \chi T_{tt}^{(0)} \\ H_{ij}^{(2)} = f'(0)R_{ij}^{(2)} + \left[\frac{f'(0)}{2}R^{(2)} + f''(0)\Delta R^{(2)} \right] \delta_{ij} - f''(0)R_{,ij}^{(2)} = 0 \\ H^{(2)} = -3f''(0)\Delta R^{(2)} - f'(0)R^{(2)} = \chi T^{(0)} \end{array} \right.$$

At third order

$$H_{ti}^{(3)} = f'(0)R_{ti}^{(3)} - f''(0)R^{(2)}_{,ti} = \chi T_{ti}^{(1)}$$

And at fourth order ...

Newtonian and Post-Newtonian of $f(R)$ -Gravity in isotropic coordinates



$$\left\{ \begin{aligned}
 H_{tt}^{(4)} &= f'(0)R_{tt}^{(4)} + f''(0)R^{(2)}R_{tt}^{(2)} - \frac{f'(0)}{2}R^{(4)} - \frac{f'(0)}{2}g_{tt}^{(2)}R^{(2)} - \frac{f''(0)}{4}R^{(2)^2} \\
 &\quad - f''(0) \left[g_{mn,m}^{(2)}R^{(2)}_{,n} + \Delta R^{(4)} + g_{tt}^{(2)}\Delta R^{(2)} + g_{mn}^{(2)}R^{(2)}_{,mn} - \frac{1}{2}\nabla g_{mm}^{(2)} \cdot \nabla R^{(2)} \right] \\
 &\quad - f'''(0) \left[|\nabla R^{(2)}|^2 + R^{(2)}\Delta R^{(2)} \right] = \mathcal{X} T_{tt}^{(2)} \\
 \\
 H^{(4)} &= -3f''(0)\Delta R^{(4)} - f'(0)R^{(4)} - 3f'''(0) \left[|\nabla R^{(2)}|^2 + R^{(2)}\Delta R^{(2)} \right] \\
 &\quad + 3f''(0) \left[R_{,tt}^{(2)} - g_{mn}^{(2)}R_{,mn}^{(2)} - \frac{1}{2}\nabla(g_{tt}^{(2)} - g_{mm}^{(2)}) \cdot \nabla R^{(2)} - g_{mn,m}^{(2)}R_{,n}^{(2)} \right] = \mathcal{X} T^{(2)}
 \end{aligned} \right.$$

Newtonian and Post-Newtonian of $f(R)$ -Gravity in isotropic coordinates



The solutions can be formulated by using the Green functions method (**the Newtonian level is linear!!!**). The Ricci scalar is

$$R^{(2)}(t, \mathbf{x}) = \frac{m^2 \mathcal{X}}{f'(0)} \int d^3 \mathbf{x}' \mathcal{G}(\mathbf{x}, \mathbf{x}') T^{(0)}(t, \mathbf{x}') \quad m^2 \doteq -\frac{f'(0)}{3f''(0)}$$

where $G(\mathbf{x}, \mathbf{x}')$ is the Green function of field operator $\Delta - m^2$.

The metric components are

$$g_{tt}^{(2)}(t, \mathbf{x}) = -\frac{\mathcal{X}}{2\pi f'(0)} \int d^3 \mathbf{x}' \frac{T_{tt}^{(0)}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{4\pi} \int d^3 \mathbf{x}' \frac{R^{(2)}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} - \frac{2}{3m^2} R^{(2)}(t, \mathbf{x})$$

$$g_{ti}^{(3)}(t, \mathbf{x}) = -\frac{\mathcal{X}}{2\pi f'(0)} \int d^3 \mathbf{x}' \frac{T_{ti}^{(1)}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \frac{1}{6\pi m^2} \frac{\partial}{\partial t} \int d^3 \mathbf{x}' \frac{\nabla_{i'} R^{(2)}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

$$g_{ij}^{(2)}(t, \mathbf{x}) = \left[\frac{1}{4\pi} \int d^3 \mathbf{x}' \frac{R^{(2)}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \frac{2}{3m^2} R^{(2)}(t, \mathbf{x}) - \frac{1}{6\pi m^2} \frac{1}{|\mathbf{x}|^3} \int_{\Omega_{|\mathbf{x}|}} d^3 \mathbf{x}' R^{(2)}(t, \mathbf{x}') \right] \delta_{ij}$$

$$+ \left[\frac{1}{2\pi m^2 |\mathbf{x}|^3} \int_{\Omega_{|\mathbf{x}|}} d^3 \mathbf{x}' R^{(2)}(t, \mathbf{x}') - \frac{2}{3m^2} R^{(2)}(t, \mathbf{x}) \right] \frac{x_i x_j}{|\mathbf{x}|^2}$$

And at fourth order ...



Newtonian and Post-Newtonian of $f(R)$ -Gravity in isotropic coordinates

$$R^{(4)}(t, \mathbf{x}) = \int d^3 \mathbf{x}' \mathcal{G}(\mathbf{x}, \mathbf{x}') \left\{ \frac{m^2 \chi}{f'(0)} T^{(2)}(t, \mathbf{x}') - g_{mn,m}^{(2)}(t, \mathbf{x}') R_{,n}^{(2)}(t, \mathbf{x}') - g_{mn}^{(2)}(t, \mathbf{x}') R_{,mn}^{(2)}(t, \mathbf{x}') \right. \\ \left. + R_{,tt}^{(2)}(t, \mathbf{x}') - \frac{m^2}{\mu^4} \left[|\nabla_{\mathbf{x}'} R^{(2)}(t, \mathbf{x}')|^2 + R^{(2)}(t, \mathbf{x}') \Delta_{\mathbf{x}'} R^{(2)}(t, \mathbf{x}') \right] \right. \\ \left. - \frac{1}{2} \nabla_{\mathbf{x}'} \left[g_{tt}^{(2)}(t, \mathbf{x}') - g_{mm}^{(2)}(t, \mathbf{x}') \right] \cdot \nabla_{\mathbf{x}'} R^{(2)}(t, \mathbf{x}') \right\}$$

$$\mu^4 \doteq -\frac{f'(0)}{3f'''(0)}$$

$$g_{tt}^{(4)}(t, \mathbf{x}) = \int d^3 \mathbf{x}' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \left\{ -\frac{\chi T_{tt}^{(2)}(t, \mathbf{x}')}{2\pi f'(0)} + \frac{1}{6\pi\mu^4} \left[|\nabla R^{(2)}(t, \mathbf{x}')|^2 + R^{(2)}(t, \mathbf{x}') \Delta R^{(2)}(t, \mathbf{x}') \right] \right. \\ \left. + \frac{1}{4\pi} \left[g_{mn}^{(2)}(t, \mathbf{x}') g_{tt,mn}^{(2)}(t, \mathbf{x}') - g_{tt,tt}^{(2)}(t, \mathbf{x}') - |\nabla_{\mathbf{x}'} g_{tt}^{(2)}(t, \mathbf{x}')|^2 - R^{(4)}(t, \mathbf{x}') - g_{tt}^{(2)}(t, \mathbf{x}') R^{(2)}(t, \mathbf{x}') \right] \right. \\ \left. + \frac{1}{6\pi m^2} \left[\frac{R^{(2)2}(t, \mathbf{x}')}{4} - \frac{R^{(2)}(t, \mathbf{x}') \Delta g_{tt}^{(2)}(t, \mathbf{x}')}{2} + g_{mn,m}^{(2)}(t, \mathbf{x}') R_{,n}^{(2)}(t, \mathbf{x}') + \Delta R^{(4)}(t, \mathbf{x}') \right. \right. \\ \left. \left. + g_{tt}^{(2)}(t, \mathbf{x}') \Delta R^{(2)}(t, \mathbf{x}') + g_{mn}^{(2)}(t, \mathbf{x}') R_{,mn}^{(2)}(t, \mathbf{x}') - \frac{1}{2} \nabla g_{mm}^{(2)}(t, \mathbf{x}') \cdot \nabla R^{(2)}(t, \mathbf{x}') \right] \right\} \quad ($$

Newtonian and Post-Newtonian of $f(R)$ -Gravity in isotropic coordinates



The possible choices of the Green functions for spherically symmetric systems are the following

$$\mathcal{G}(\mathbf{x}, \mathbf{x}') = \begin{cases} -\frac{1}{4\pi} \frac{e^{-m|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} & \text{if } m^2 > 0 \\ C_1 \frac{e^{-im|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} + C_2 \frac{e^{im|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} & \text{if } m^2 < 0 \end{cases}$$

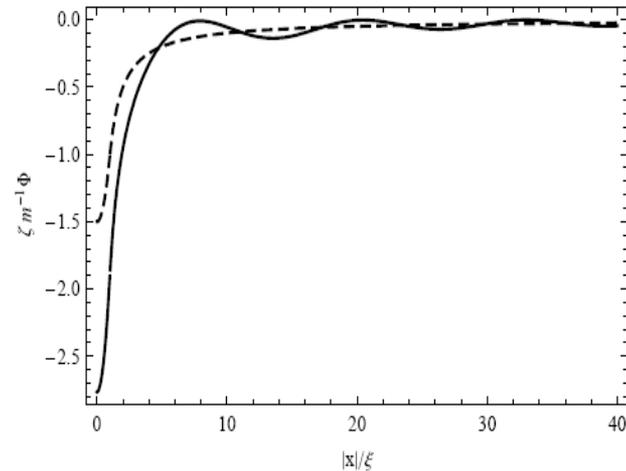
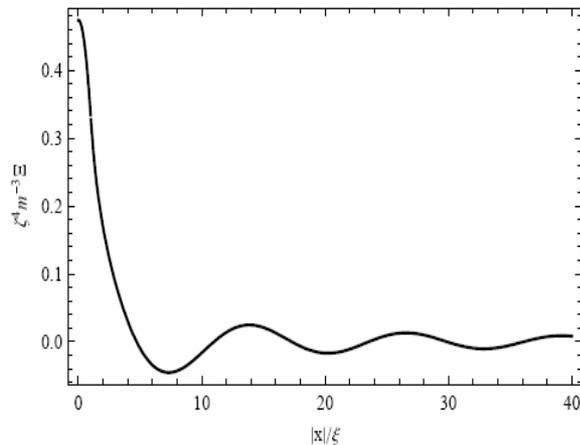
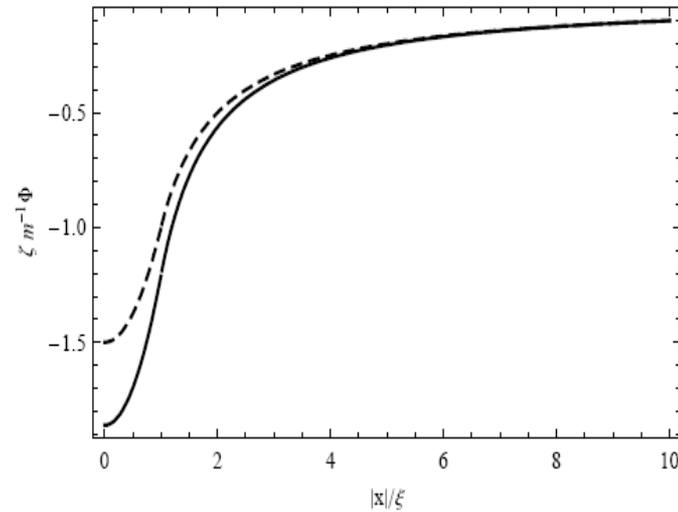
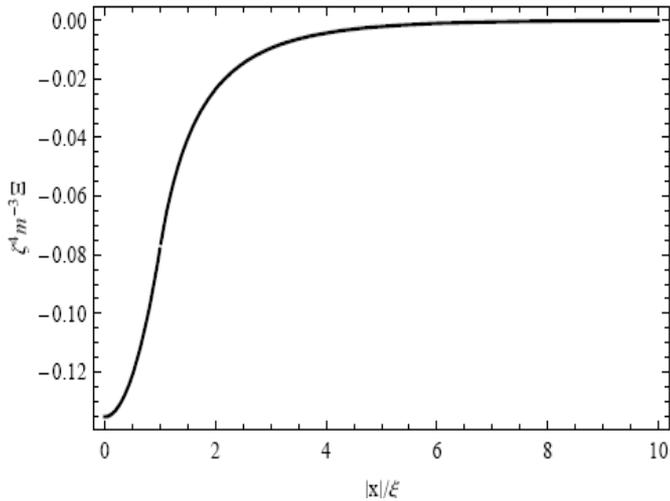
In the Newtonian limit of GR, the equation for the gravitational potential, generated by a point-like source is

$$\Delta_{\mathbf{x}} \mathcal{G}_{New.mech.}(\mathbf{x} - \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \quad \mathbf{g}_{New.mech.} = -\frac{GM(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} = -GM\nabla_{\mathbf{x}} \mathcal{G}_{New.mech.}(\mathbf{x} - \mathbf{x}')$$

And the Gauss theorem holds $\int_{\Sigma} d\Sigma \mathbf{g}_{New.mech.} \cdot \hat{n} \propto M$

But in $f(R)$ -Gravity $\int_{\Sigma} d\Sigma \mathbf{g}_{New.mech.} \cdot \hat{n} \propto M_{\Sigma}$ **Then the Gauss theorem is not verified!!!**

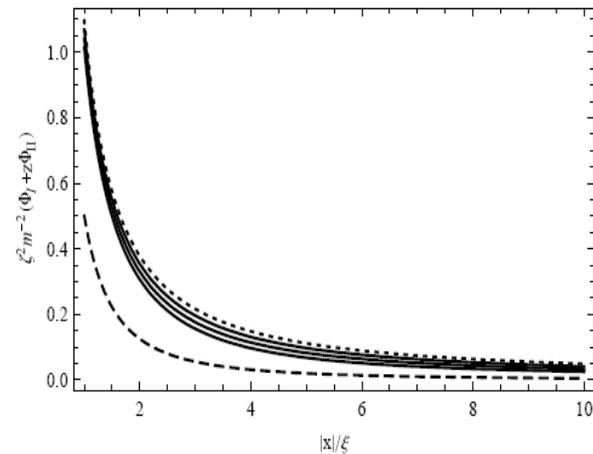
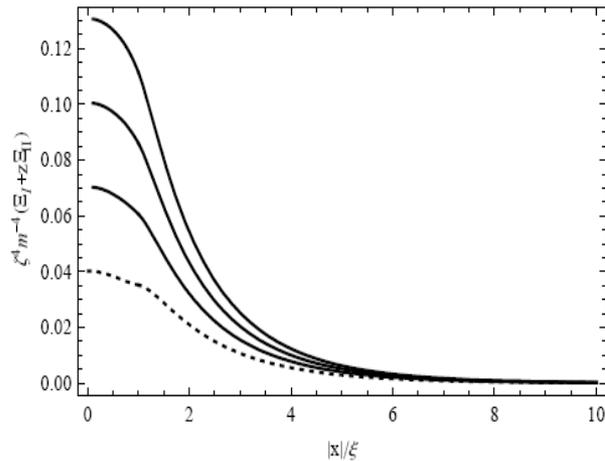
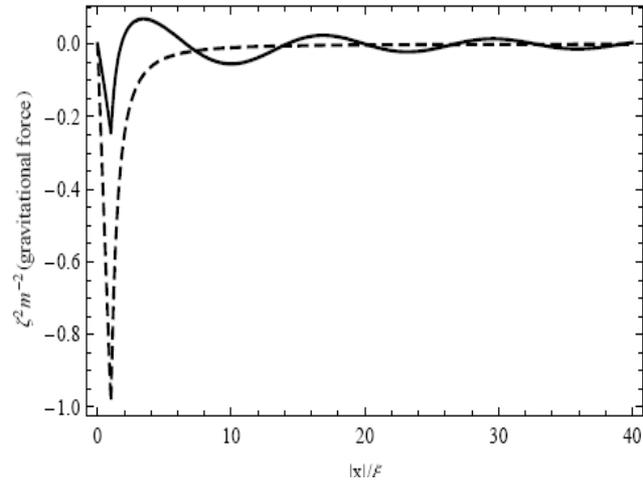
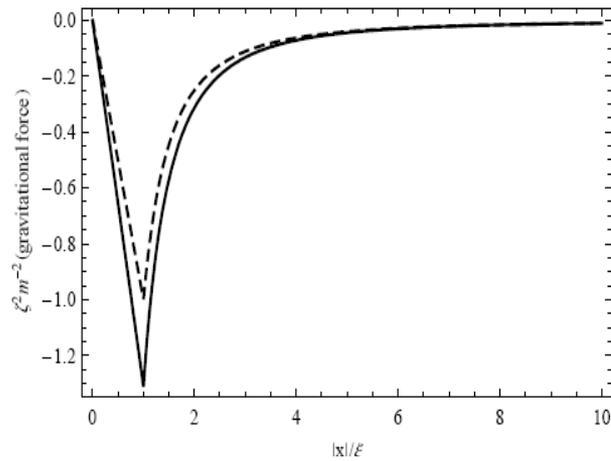
The potential and the Ricci scalar generated by ball-like source



The potential and the Ricci scalar generated by ball-like source



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The Newtonian limit of FOG

The field equations are

$$\left\{ \begin{array}{l} (\Delta - m_2^2)R_{tt}^{(2)} + \left[\frac{m_2^2}{2} - \frac{m_1^2 + 2m_2^2}{6m_1^2} \Delta \right] X^{(2)} = -\frac{m_2^2 \mathcal{X}}{f_X(0)} T_{tt}^{(0)} \\ (\Delta - m_2^2)R_{ij}^{(2)} + \left[\frac{m_1^2 - m_2^2}{3m_1^2} \partial_{ij}^2 - \left(\frac{m_2^2}{2} - \frac{m_1^2 + 2m_2^2}{6m_1^2} \Delta \right) \delta_{ij} \right] X^{(2)} = 0 \\ (\Delta - m_1^2)X^{(2)} = \frac{m_1^2 \mathcal{X}}{f_X(0)} T^{(0)} \end{array} \right. \quad \left\{ \begin{array}{l} m_1^2 \doteq -\frac{f_X(0)}{3f_{XX}(0) + 2f_Y(0) + 2f_Z(0)} \\ m_2^2 \doteq \frac{f_X(0)}{f_Y(0) + 4f_Z(0)} \end{array} \right.$$

and the general solutions are $X^{(2)}(t, \mathbf{x}) = \frac{m_1^2 \mathcal{X}}{f_X(0)} \int d^3 \mathbf{x}' \mathcal{G}_1(\mathbf{x}, \mathbf{x}') T^{(0)}(t, \mathbf{x}')$

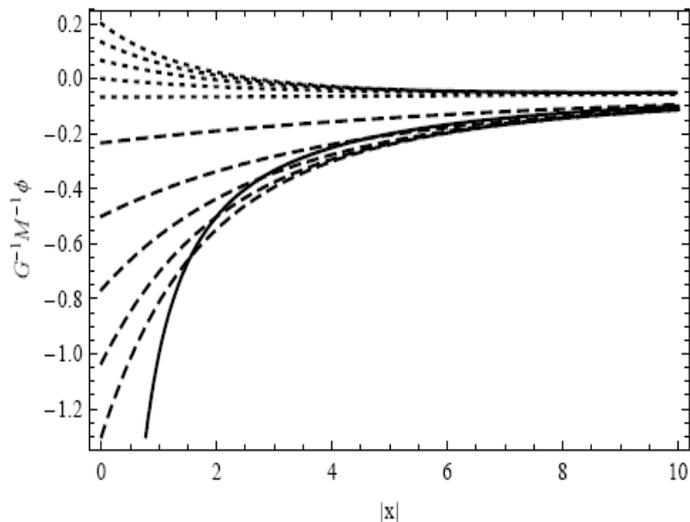
$$g_{tt}^{(2)}(t, \mathbf{x}) = \frac{1}{2\pi} \int d^3 \mathbf{x}' d^3 \mathbf{x}'' \frac{\mathcal{G}_2(\mathbf{x}', \mathbf{x}'')}{|\mathbf{x} - \mathbf{x}'|} \left[\frac{m_2^2 \mathcal{X}}{f_X(0)} T_{tt}^{(0)}(t, \mathbf{x}'') - \frac{(m_1^2 + 2m_2^2) \mathcal{X}}{6f_X(0)} T^{(0)}(t, \mathbf{x}'') + \frac{m_2^2 - m_1^2}{6} X^{(2)}(t, \mathbf{x}'') \right]$$

$$g_{ij}^{(2)} = 2\Psi \delta_{ij} = -\frac{\delta_{ij}}{2\pi} \int d^3 \mathbf{x}' d^3 \mathbf{x}'' \frac{\mathcal{G}_2(\mathbf{x}', \mathbf{x}'')}{|\mathbf{x} - \mathbf{x}'|} \left(\frac{m_2^2}{2} - \frac{m_1^2 + 2m_2^2}{6m_1^2} \Delta_{\mathbf{x}''} \right) X^{(2)}(\mathbf{x}'')$$

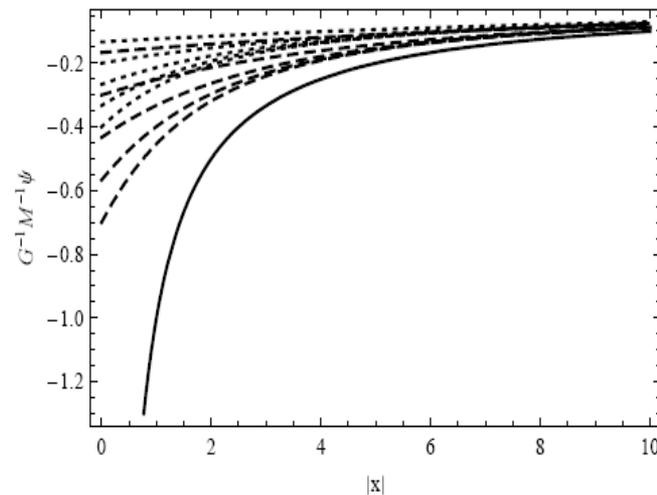


The solutions at Newtonian limit of FOG

In the case of pointlike source of mass M



$$\left\{ \begin{array}{l} X^{(2)} = -\frac{r_g m_1^2 e^{-m_1|x|}}{f_X(0)|x|} \\ \Phi = -\frac{GM}{f_X(0)} \left[\frac{1}{|x|} + \frac{1}{3} \frac{e^{-m_1|x|}}{|x|} - \frac{4}{3} \frac{e^{-m_2|x|}}{|x|} \right] \\ \Psi = -\frac{GM}{f_X(0)} \left[\frac{1}{|x|} - \frac{1}{3} \frac{e^{-m_1|x|}}{|x|} - \frac{2}{3} \frac{e^{-m_2|x|}}{|x|} \right] \end{array} \right.$$



For a physical interpretation of potential we must have

$$f_{XX}(0) + f_Y(0) + 2f_Z(0) < 0$$



Hydrostatic equilibrium

The Lane-Emden equation

$$\frac{d^2 w(z)}{dz^2} + \frac{2}{z} \frac{dw(z)}{dz} + w(z)^n = \frac{m\xi_0}{8} \frac{1}{z} \int_0^{\xi/\xi_0} dz' z' \left\{ e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|} \right\} w(z')^n$$

$$w_{f(n)}(z) = \left[\frac{3}{2\xi} + \frac{1}{m^2\xi^3} - \frac{e^{-m\xi}(1+m\xi)}{m^2\xi^3} \right]^{-1} \left[\frac{3}{2\xi} + \frac{1}{m^2\xi^3} - \frac{\xi_0^2 z^2}{2\xi^3} - \frac{e^{-m\xi}(1+m\xi)}{m^2\xi^3} \frac{\sinh m\xi_0 z}{m\xi_0 z} \right]$$

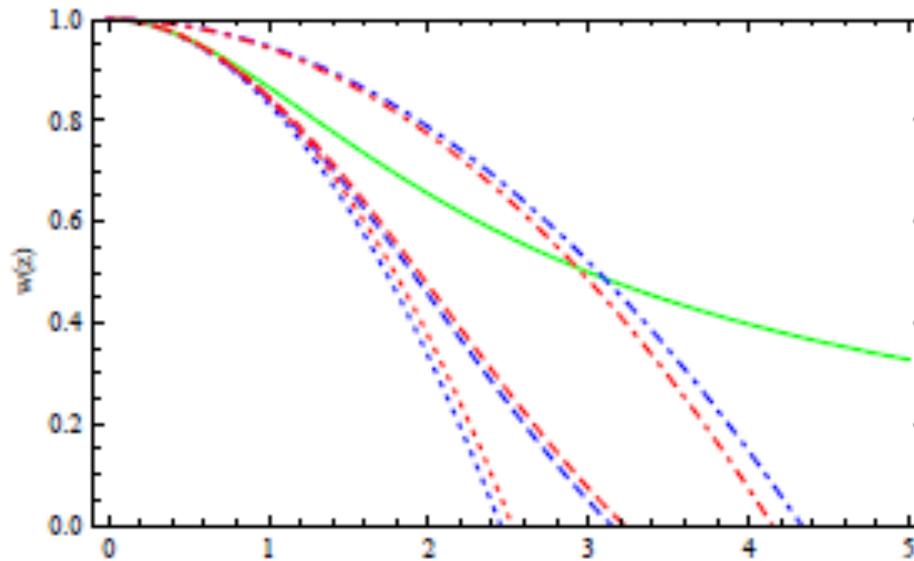
$$w_{f(n)}^{(0)}(z) = 1 - \frac{z^2}{8} + \frac{(1+m\xi)e^{-m\xi}}{4m^2\xi_0^2} \left[1 - \frac{\sinh m\xi_0 z}{m\xi_0 z} \right]$$

$$w_{f(n)}^{(1)}(z) \sim \frac{\sin z}{z} \left\{ 1 + \frac{m^2\xi_0^2}{8(1+m^2\xi_0^2)} \left[1 + \frac{2e^{-m\xi}}{1+m^2\xi_0^2} (\cos \xi/\xi_0 + m\xi_0 \sin \xi/\xi_0) \right] \right\} \\ - \frac{m^2\xi_0^2}{8(1+m^2\xi_0^2)} \left[\frac{2e^{-m\xi}}{1+m^2\xi_0^2} (\cos \xi/\xi_0 + m\xi_0 \sin \xi/\xi_0) \frac{\sinh m\xi_0 z}{m\xi_0 z} + \cos z \right]$$



Hydrostatic equilibrium

The radial behaviour of gravitational potential



Post-Minkowskian limit of f -gravity: gravitational waves in f -gravity



The post-Minkowskian limit is recovered if we consider **only the weak field** hypothesis. In this case the time derivative is of the same order of space derivative. The Laplacian, in the PPN formalism, is replaced by the d'Alembertian.

The work hypothesis:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

The field equations in the harmonic gauge become

$$\begin{cases} \square_{\eta} \tilde{h}_{\mu\nu} + \frac{1}{3\lambda^2} (\eta_{\mu\nu} \square_{\eta} - \partial_{\mu\nu}^2) \square_{\eta} \tilde{h} = -\frac{2\mathcal{X}}{f_1} T_{\mu\nu}^{(0)} \\ \square_{\eta} \tilde{h} + \frac{1}{\lambda^2} \square_{\eta}^2 \tilde{h} = -\frac{2\mathcal{X}}{f_1} T^{(0)} \end{cases}$$

The solution in term of Fourier transform is

$$h_{\mu\nu}(k) = \frac{2\mathcal{X}}{f_1} \frac{S_{\mu\nu}^{(0)}(k)}{k^2} - \frac{\mathcal{X}}{3f_1} \frac{k^2 \eta_{\mu\nu} + 2k_{\mu} k_{\nu}}{k^2(k^2 - \lambda^2)} S^{(0)}(k) \quad \square_{\eta} h_{\mu\nu}(x) = -\frac{2\mathcal{X}}{f_1} \left[S_{\mu\nu}^{(0)}(x) + \Sigma_{\mu\nu}^{\lambda}(x) \right]$$



Energy-momentum tensor of f -gravity

As in GR, we try to define a Energy-momentum tensor. In f -gravity it is possible to write

$$t^\lambda{}_\alpha = f' \left\{ \left[\frac{\partial R}{\partial g_{\rho\sigma,\lambda}} - \frac{1}{\sqrt{-g}} \partial_\xi \left(\sqrt{-g} \frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}} \right) \right] g_{\rho\sigma,\alpha} + \frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\xi\alpha} \right\} \\ - f'' R_{,\xi} \frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\alpha} - \delta_\alpha^\lambda f .$$

from which we have

$$[\sqrt{-g}(t^\lambda{}_\alpha + 2\mathcal{X} T^\lambda{}_\alpha)],\lambda = 0$$

in terms of perturbation:

$$t^\lambda{}_\alpha \sim f'(0)t^\lambda{}_{\alpha|_{\text{GR}}} + f''(0) \left\{ (h^{\rho\sigma}{}_{,\rho\sigma} - \square h) [h^{\lambda\xi}{}_{,\xi\alpha} - h^{\lambda\alpha}{}_{,\alpha} - \frac{1}{2} \delta_\alpha^\lambda (h^{\rho\sigma}{}_{,\rho\sigma} - \square h)] \right. \\ \left. - h^{\rho\sigma}{}_{,\rho\sigma\xi} h^{\lambda\xi}{}_{,\alpha} + h^{\rho\sigma}{}_{,\rho\sigma}{}^\lambda h_{,\alpha} + h^{\lambda\xi}{}_{,\alpha} \square h_{,\xi} - \square h^{\lambda\alpha} h_{,\alpha} \right\} ,$$

Perspectives



Experimental testing of all results:

- 1) Galaxies rotation curves
- 2) Solar System experiments
- 3) Effective Theories from Unification Schemes
- 4) GW experiments

List of papers



Articles published in refereed journals

- *Hydrostatic Equilibrium and Stellar Structure in $f(R)$ -Gravity* – S. Capozziello, M. De Laurentis, S. D. Odintsov, A. Stabile – Physical Review D **83**, 064004 (2011), pp 6
- *The Most General Fourth Order Theory of Gravity at Low Energy* – A. Stabile – Physical Review D **82**, 124026 (2010), pp 7
- *The Post-Newtonian Limit of $f(R)$ – Gravity in the Harmonic Gauge* – A. Stabile – Physical Review D **82**, 064021 (2010), pp 12
- *Axially symmetric solutions in $f(R)$ – Gravity* - S. Capozziello, M. De Laurentis, A. Stabile – Classical and Quantum Gravity **27**, 165008 (2010), pp 16
- *The post Minkowskian limit in $f(R)$ – Gravity* - S. Capozziello, A. Stabile, A. Troisi – International Journal of Theoretical Physics **49**, 1251 (2010), pp 12
- *Comparing scalar-tensor gravity and $f(R)$ - Gravity in the Newtonian limit* – S. Capozziello, A. Stabile, A. Troisi – Physics Letters B **686**, 79 (2010), pp 5
- *The Newtonian limit of metric Gravity theories with quadratic Lagrangians* – S. Capozziello, A. Stabile – Classical and Quantum Gravity **26**, 085019 (2009), pp 22

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Articles published in refereed journals

- *A new general solution in the Newtonian Limit of $f(R)$ – Gravity* - S. Capozziello, A. Stabile, A. Troisi, *Modern Physics Letters A* **24**, No 9 659 (2009), pp 7
- *Spherical symmetry in $f(R)$ - Gravity* - S. Capozziello, A. Stabile, A. Troisi, - *Classical and Quantum Gravity* **25**, 085004 (2008), pp 14
- *Newtonian limit of $f(R)$ - Gravity* - S. Capozziello, A. Stabile, A. Troisi, *Physical Review D* **76**, 104019 (2007), pp 12
- *Spherically symmetric solutions in $f(R)$ - Gravity via Noether symmetry approach* - S. Capozziello, A. Stabile, A. Troisi, *Classical and Quantum Gravity* **24**, 2153 (2007), pp 14
- *Fourth-order gravity and experimental constraints on Eddington parameters* – S. Capozziello, A. Stabile, A. Troisi, *Modern Physics Letters A* **21**, 2291 (2006), pp 11

List of papers



Review articles published in refereed journals

- *The Weak Field Limit of Fourth Order Gravity* – S. Capozziello, A. Stabile – Classical and Quantum Gravity, chapter 2, 1 Nova Science Publishers 2010, Inc. ISBN 978-1-61122-957-8, pp 81

Articles in refereed conference proceedings

- *E-cloud Map Formalism: an Analytical Expression for Quadratic Coefficient* – T. Demma, S. Petracca, A. Stabile – Beam Dynamics and Electromagnetic Fields D05 Instabilities, Processes, Impedances, Countermeasures TUPD037 International Particle Accelerator Conference 2010 (IPAC 2010) - ISBN 978-92-9083-352-9
- *Maps for Electron Clouds: Application to LHC Conditioning* – T. Demma, R. Cimino, A. Drago, S. Petracca, A. Stabile – Beam Dynamics and Electromagnetic Fields D04 Processes, Impedances, Countermeasures - Particle Accelerator Conference 2009 - PAC09 FR5RFP071

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Articles in press in refereed conference proceedings

- *A Physical Interpretation of the Cubic Map Coefficients Describing the Electron Cloud Evolution* – T. Demma, S. Petracca, A. Stabile – XVIII National conference of Electromagnetism (RiNEm 2010) – ID EDAP2 - [ArXiv: 1011.1682v1](#) [physics.acc-ph]
- *Weak Field Approach in $f(R)$ - Gravity* – A. Stabile, S. Capozziello – 54th National Congress of Italian Astronomical Society (SAIt 2010) – [ArXiv: 1010.0099v1](#) [gr-qc]

Presentations as speaker

- *Weak Field Limit of Fourth Order Gravity* – A. Stabile*, S. Capozziello – XCVI Congress of Italian society of Physics
- *Dinamica dell' Electron Cloud: Calcolo dei Coefficienti della Mappa* – T. Demma, S. Petracca, A. Stabile* - XCVI Congress of Italian society of Physics
- *Maps for Electron Cloud in LHC* - T. Demma, S. Petracca, A. Stabile* – XCV Congress of Italian society of Physics

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Posters

- *E-cloud Map Formalism: an Analytical Expression for Quadratic Coefficient* – T. Demma*, S. Petracca, A. Stabile – International Particle Accelerator Conference 2010 (IPAC10)
- *Weak Field Approach in $f(R)$ -Gravity* – A. Stabile, S. Capozziello* – 54° Congresso Nazionale della Società Astronomica Italiana (SAIt 2010)
- *Maps for Electron Cloud in LHC* – S. Petracca, A. Stabile*, T. Demma - “European Researchers night 2009” at University of Sannio – Benevento, Italy
- *Maps for Electron Clouds: Application to LHC Conditioning* – T. Demma*, R. Cimino, A. Drago, S. Petracca, A. Stabile – Particle Accelerator Conference 2009 (PAC09)

(*): Speaker.