In recent years, the effort to give a physical explanation to the today observed cosmic acceleration has attracted a good amount of interest in Fourth Order Gravity (FOG) considered as a viable mechanism to explain the cosmic acceleration by extending the geometric sector of field equations without the introduction of dark matter and dark energy. At fundamental level, several efforts have been aimed towards the unification of gravity with the other interactions of physics (like Electromagnetism), assuming GR as the only fundamental theory capable of explaining the gravitational interaction. The failure of such attempts led to the common belief that GR had to be revised in the ultraviolet limit in order to address issues like quantization and renormalization. There are only some impacts of the several physical and mathematical motivations to enlarge GR in more general approaches. Other issues come from astrophysics. The observed Pioneer anomaly problem can be framed into the same approach and then, apart the cosmology and quantum field theory, it is of utmost importance to directly address the way field equations work in the whole sector of field equations and observations. (ii) A second set of criteria are self-consistent and comprehensive physics phenomena as acceleration matter, or dark matter at Galactic scales. It is straightforward to extend the field equations by adding several more fundamental degrees of freedom and several modified postulates focused on the role of modifying the gravitational Lagrangian in its complete metric framework, leading to the fourth order and higher order field equations. Such an approach has become a sort of pendant in the study of gravitational interaction considering, essentially, adding higher order curvature invariants and explicitly or non-explicitly coupled scalar fields into dynamics which came out from the effective action of unification or quantum gravity theory. The idea to extend Einstein’s theory of gravitation is fruitful and economic also with respect to several attempts which, in spite of progressing by testing new and most of them, unsuited ingredients in order to give self-consistent pictures of physics. The idea observed accelerated expansion of the Hubble flow and the missing matter at astrophysical scales are primarily excluded in these considerations. Both the issues could be solved by changing the gravitational sector i.e. the field equations. The philosophy is alternative to add new cosmic fluids (new field sectors) which should inhabit a new kind of structures (dark matter) or to accelerated dynamic (dark energy) thanks to exotic equations of state. In particular: removing the hypothesis that the Lagrangian has to be a linear function of the Ricci scalar or, like in the Hilbert-Einstein formulation, one can take into account an effective action in which the gravitational Lagrangian includes other scalar invariants. Due to the increased complexity of the field equations, the many-body of theoretical works dealt with the effort to achieve some formally equivalent theories which could be handed in a simpler way. We want to address the general problem of the weak field limit for theories of gravity where higher order curvature invariants are present. In particular, we deal with theories where Riemann tensor, Ricci tensor, and Ricci scalar are considered in the effective action.

The Universe evolution is characterized by different phases of expansion:

1. Matter
2. Dark Matter
3. Ordinary Matter
4. Radiation

There is a fundamental issue: the cosmological observations and cosmology pressing the breakdown of General Relativity at large scales?

Newtonian limit of FOG

The Lagrangian of the massive particle embedded in the gravitational field is proportional to the "reductive distance". The gravitational potential is obtained by applying the variational principle.

The approximation level of field equations is driven by an expansion of the powers of (η/η0) which is the light speed and η is integer constant.

At least level (n = 1) we have the Newtonian Mechanics

First coordinates are η = 1 and η = 2

Then the Lagrangian can be expressed as

and the metric tensor assumes the usual form

For n = 2 we have the so-called Post-Newtonian Mechanics

By using the Bianchi’s identities and introducing two scale lengths m1, m2, the field equations (Geometrization limit) are

The general solution is then found as follows:

The "effective Lagrangian" of the FOG in the Newtonian level

By using the Gauss – Bonnet invariant IG2 = 3 – D2, we have the more simple but in the same time the most complete expression of the previous solutions. At Newtonian level and FOG collapses in an "effective lagrangian".

In the case of pointlike sources the equations are

Identify in the field equation a. b. c. d. e. f. g. h. i.

a. b. c. d. e. f. g. h. i.

There is a fundamental difference among the GR and FOG validity or not of Gauss theorem

Field equations approach Post-Newtonian approach

\[ \frac{d}{dt}v = -\frac{\alpha}{\beta} g^{\alpha\beta} \frac{\partial L_{\text{GR}}}{\partial \dot{\alpha}} \]

\[ \frac{d}{dt}v = -\frac{\alpha}{\beta} g^{\alpha\beta} \frac{\partial L_{\text{FOG}}}{\partial \dot{\alpha}} \]

We find a relation between the metric potentials: \( \alpha = \beta \)

The solution, mathematically exact, for a ⋅ b = 1, is

The BIRKHOFF THEOREM in (R)-Gravity... but also in FOG

At first order the field equations states the relation between the GR components and the FOG components.

In fact, generally, if the "a" component is time-independent and the "b" component is the product of two functions (one time and one other of the space) the Ricci scalar is time independent. Indeed exists every a realization time gives a two metric components.

But it is not verified the Birkhoff theorem in FOG... it works only at Newtonian level.

The Noether Symmetry approach to f(R)-Gravity

We worked out an exact covariant solutions in FOG. In order to develop such an approach, we need to adopt a point-like Lagrangian from the general action. Such a Lagrangian can be obtained by imposing the spherical symmetry in the field action. As a consequence, the infinite number of degrees of freedom of the original field theory will be reduced to a finite number.

The Euler-Lagrange equations are in terms of the function A, B, M, H, f. The field equation for H corresponds to the constraint among the configuration coordinates.

It is worth noting that the Hessian determinat is zero! The point-like Lagrangian does not depend on B. In other words, B does not contribute to dynamics and its equation has to be considered as a further constraint equation. The field-equations approach and the point-like Lagrangian approach differ since the symmetry, in our case the spherical one, can be imposed whether in the field equations, after standard variation with respect to the metric, or directly into the Lagrangian, which becomes point-like.

The Noether Symmetry approach we find a consistent for a power law f(R) = R1/2.

The Galactic Rotation Curves

Generally the superposition principle is not valid in FOG but, we are at investigating this limit (linear field equations). There we give the gravitational potential and the geometric curve relation, respectively, as

There is a fundamental difference among the GR and FOG validity or not of Gauss theorem

\[ \frac{d}{dt}v = -\frac{\alpha}{\beta} g^{\alpha\beta} \frac{\partial L_{\text{GR}}}{\partial \dot{\alpha}} \]

\[ \frac{d}{dt}v = -\frac{\alpha}{\beta} g^{\alpha\beta} \frac{\partial L_{\text{FOG}}}{\partial \dot{\alpha}} \]

The Gravitational Lensing by Extended Matter Source

The starting point is

\[ \phi = \beta \]

The metric potentials

The geodetic motion

In the metric potentials are equal, but in FOG we have

L = \beta \phi + \alpha \phi^2 + \gamma \phi^3 + \epsilon \phi^4

The deflection \( \epsilon \) \( \epsilon \) is given by

\[ \epsilon = \frac{\beta}{1 + \beta} \]

The effect in which the planar and point-like source becomes

We need DM component!

Unknown!!

The content of the universe is, up today, absolutely unknown for its largest part. The situation is very "DARK" while the observations are somehow good!