INFN Istituto Nazionale di Fisica Nucleare

EXTENDED THEORIES OF GRAVITY

A. Stabile

www.arturostabile.com

Università degli Studi del Sannio

SIGRAV

In recent years, the effort to give a physical explanation to the today observed cosmic acceleration has attracted a good amount of interest in Fourth Order Gravity (FOG) considered as a viable mechanism to explain the cosmic acceleration by extending the geometric sector of field equations without the introduction of dark matter and dark energy. At fundamental level, several efforts have been aimed towards the unification of gravity with the other interactions of physics, like Electromagnetism, assuming GR as the only fundamental theory capable of explaining the gravitational interaction. The failure of such attempts led to the common belief that GR had to be revised in the ultraviolet limit in order to address issues like quantization and renormalization. These are only some aspects of the several physical and mathematical motivations to enlarge GR to more general approaches. Other issues come from astrophysics. The observed Pioneer anomaly problem can be framed into the same approach and then, apart the cosmology and quantum field theory, a systematic analysis of such theories urges at small, medium and large scales. In particular, a delicate point is to address the weak field limit of any extended theory of gravity since two main issues are extremely relevant: i) preserving the results of GR al local scales since they well fit Solar System experiments and observations; ii) enclosing in a self-consistent and comprehensive picture phenomena as anomalous acceleration or dark matter at Galactic scales. It is straightforward to extend GR to theories with additional geometric degrees of freedom and several recent proposals focused on the old idea of modifying the gravitational Lagrangian in a purely metric framework, leading to fourth-order and higher-order field equations. Such an approach has become a sort of paradigm in the study of gravitational interaction consisting, essentially, ***in adding higher order curvature invariants and minimally or non-minimally coupled scalar fields into dynamics which come out from the effective action of some unification or quantum gravity theory***. The idea to extend Einstein's theory of gravitation is fruitful and economic also with respect to several attempts which try to solve problems by adding new and, most of times, unjustified ingredients in order to give self-consistent pictures of dynamics. The today observed accelerated expansion of the Hubble flow and the missing matter at astrophysical scales are primarily enclosed in these considerations. Both the issues could be solved by changing the gravitational sector, *i.e.* the *l.h.s.* of field equations. The philosophy is alternative to add new cosmic fluids (new components in the *r.h.s.* of field equations) which should give rise to clustered structures (dark matter) or to accelerated dynamics (dark energy) thanks to exotic equations of state. In particular, relaxing the hypothesis that gravitational Lagrangian has to be a linear function of the Ricci curvature scalar *R*, like in the Hilbert-Einstein formulation, one can take into account an effective action where the gravitational Lagrangian includes other scalar invariants. Due to the increased complexity of the field equations, the main body of theoretical works dealt with the effort to achieve some formally equivalent theories which could be handled in a simpler way. We want to address the general problem of the weak field limit for theories of gravity where higher order curvature invariants are present. In particular, we deal with theories where Riemann tensor, Ricci tensor, and Ricci scalar are considered in the extended action. |

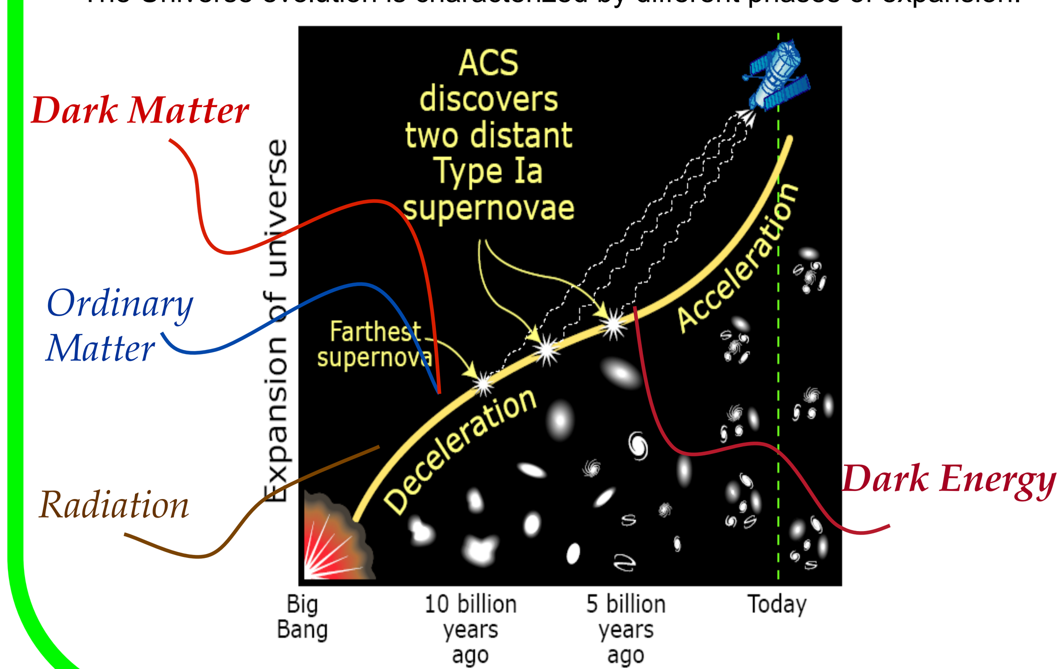
COMPOSITION OF THE UNIVERSE

MATERIAL	REPRESENTATIVE PARTICLES	TYPICAL PARTICLE MASS OR ENERGY (ELECTRON VOLTS)	NUMBER OF PARTICLES IN OBSERVED UNIVERSE	PROBABLE CONTRIBUTION TO MASS OF UNIVERSE	SAMPLE EVIDENCE
Ordinary ("baryonic") matter	Protons, electrons	10^6 to 10^9	10^{78}	5%	Direct observation, inference from element abundances
Radiation	Cosmic microwave background photons	10^{-4}	10^{87}	0.005%	Microwave telescope observations
Hot dark matter	Neutrinos	≤ 1	10^{87}	0.3%	Neutrino measurements, inference from cosmic structure
Cold dark matter	Supersymmetric particles?	10^{11}	[illegible]	25%	Inference from galaxy dynamics
Dark energy	"Scalar" particles?	10^{-33} (assuming dark energy comprises particles)	[illegible]	70%	Supernova observations of accelerated cosmic expansion

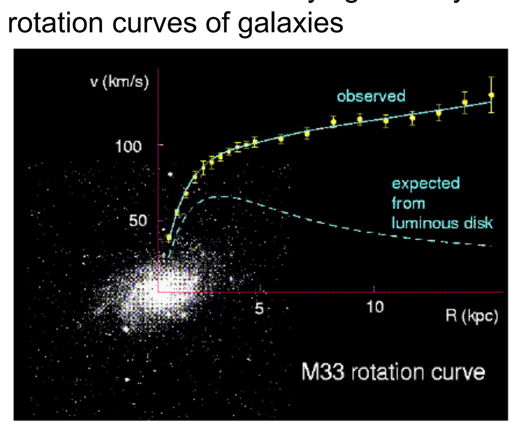
95%

Unknown!!

The Universe evolution is characterized by different phases of expansion.

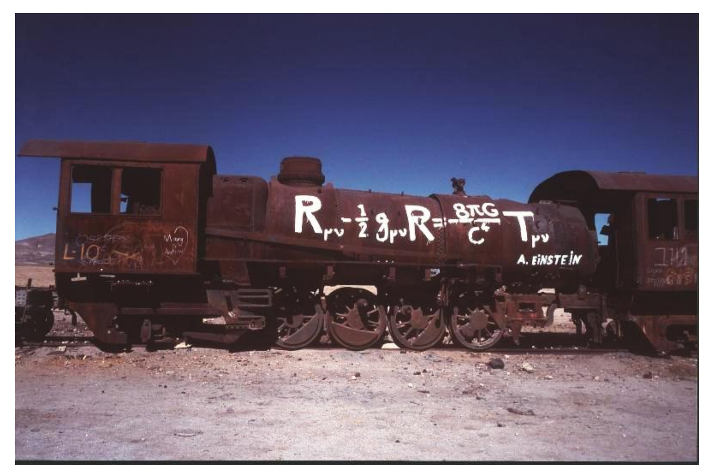


The presence of Dark Matter components has been revealed since 1933 by Zwicky as a lack in the mass content of galaxy clusters. The most peculiar effect of Dark Matter is the discover of a non-decaying velocity of rotation curves of galaxies

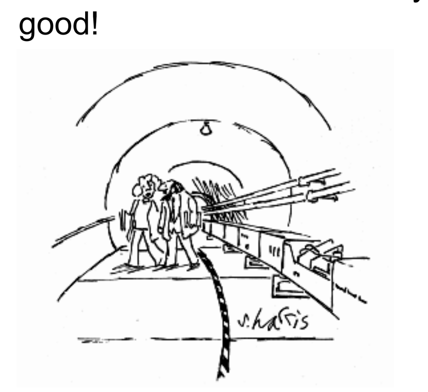


There is a fundamental issue:

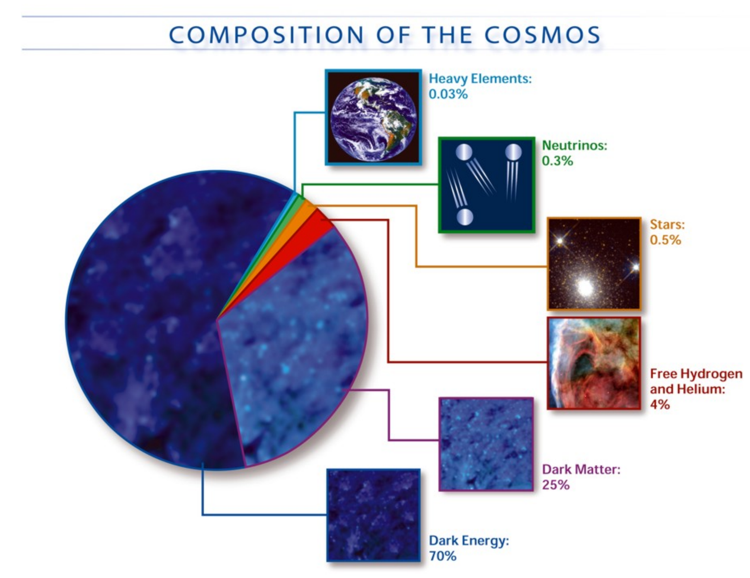
Are extragalactic observations and cosmology probing the breakdown of General Relativity at large (IR) scales?



The content of the universe is, up today, **absolutely unknown** for its largest part. *The situation is very "DARK" while the observations are extremely good!*



Dark Matter in Lab
... and if, after spending all this money, no further particles remain to discover!



FOURTH ORDER GRAVITY (FOG)

The action principle $\mathcal{A}^f = \int d^4x\sqrt{-g}\Big[f(X,Y,Z) + \mathcal{X}\mathcal{L}_m\Big]$ where $X = R$ (Ricci scalar), $Y = R_{\alpha\beta}R^{\alpha\beta}$ (Ricci tensor square) and $Z = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ (Riemann square)

By using the metric approach the field equations are

$$H_{\mu\nu} = f_X R_{\mu\nu} - \frac{f}{2}g_{\mu\nu} - f_{X;\mu\nu} + g_{\mu\nu}\Box f_X + 2f_Y R_\mu{}^\alpha R_{\alpha\nu} - 2[f_Y R^\alpha{}_{(\mu}]_{;\nu)\alpha} + \Box[f_Y R_{\mu\nu}] + [f_Y R_{\alpha\beta}]^{;\alpha\beta}g_{\mu\nu}$$
$$+2f_Z R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} - 4[f_Z R_\mu{}^{\alpha\beta}{}_\nu]_{;\alpha\beta} = \mathcal{X}T_{\mu\nu}$$

where $f_X = \frac{df}{dX}$, $f_Y = \frac{df}{dY}$, $f_Z = \frac{df}{dZ}$

The energy momentun tensor of matter is $T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$

The trace equation is $H = g^{\alpha\beta}H_{\alpha\beta} = f_X R + 2f_Y R_{\alpha\beta}R^{\alpha\beta} + 2f_Z R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 2f + \Box[3f_X + f_Y R] + 2[(f_Y + 2f_Z)R^{\alpha\beta}]_{;\alpha\beta} = \mathcal{X}T$

where the covariant derivative is given as $A^{\alpha\beta\ldots\delta}{}_{;\mu} = A^{\alpha\beta\ldots\delta}{}_{,\mu} + \Gamma^\alpha_{\sigma\mu}A^{\sigma\beta\ldots\delta} + \Gamma^\beta_{\sigma\mu}A^{\alpha\sigma\ldots\delta} + \cdots + \Gamma^\delta_{\sigma\mu}A^{\alpha\beta\ldots\sigma}$ and $\Box = \frac{\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta)}{\sqrt{-g}}$

Spherically symmetric solutions

We are interested to investigate spherically symmetric solutions. The most general symmetric solution can be written as follows

$$ds^2 = g_1(t,|\mathbf{x}|)\,dt^2 + g_2(t,|\mathbf{x}|)\,dt\,\mathbf{x}\cdot d\mathbf{x} + g_3(t,|\mathbf{x}|)(\mathbf{x}\cdot d\mathbf{x})^2 + g_4(t,|\mathbf{x}|)d|\mathbf{x}|^2$$

but we have an arbitrariness in the choice of coordinates. Then we can use also the expressions of metric (**isotropic coordinates**) $ds^2 = g_{tt}(t',r')\,dt^2 - g_{ij}(t',r')dx^idx^j$;

and the following choice (**standard coordinates**) $ds^2 = g_{tt}(t',r'')\,dt^2 - g_{rr}(t',r'')dr''^2 - r''^2 d\Omega$

We can use any mathematical transformation law between the coordinates. The field equations of space time are constructed with tensorial quantities and any diffeomorphism linking different sets of coordinates does not change the dynamics.

Newtonian limit of FOG

The Lagrangian of the massive particle embedded in the gravitational field is proportional to the "relativistic distance". $L = \left(g_{\rho\sigma}\frac{dx^\rho}{dt}\frac{dx^\sigma}{dt}\right)^{1/2} = \left(g_{tt} + 2g_{tm}v^m + g_{mn}v^mv^n\right)^{1/2}$

And the geodesic equation is obtained by applying the variational principle $\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\sigma\tau}\frac{dx^\sigma}{ds}\frac{dx^\tau}{ds} = 0$

The approximation level of field equations is driven by an expansion of the powers of $(1/c)^{2n}$ where c is the light speed and n is integer.

At lowest level (n = 1) we have the Newtonian Mechanics $\frac{d^2x^i}{dt^2} \simeq -\Gamma^i_{tt} \simeq -\frac{1}{2}\frac{\partial g_{tt}}{\partial x^i}$

First consequences are $\frac{\partial}{\partial t} \sim \mathrm{v}\cdot\nabla$ and $\frac{|\partial/\partial t|}{|\nabla|} \sim \mathcal{O}(1)$

Then the Lagrangian can be expressed as $\left(1 + g^{(2)}_{tt} + g^{(4)}_{tt} + 2g^{(3)}_{tm}v^m - \mathbf{v}^2 + g^{(2)}_{mn}v^mv^n\right)^{1/2}$

and the metric tensor assumes the usual form

$$\begin{cases} g_{tt}(t,\mathrm{x}) \simeq 1 + g^{(2)}_{tt}(t,\mathrm{x}) + g^{(4)}_{tt}(t,\mathrm{x}) + \mathcal{O}(6) \\ g_{ti}(t,\mathrm{x}) \simeq g^{(3)}_{ti}(t,\mathrm{x}) + \mathcal{O}(5) \\ g_{ij}(t,\mathrm{x}) \simeq -\delta_{ij} + g^{(2)}_{ij}(t,\mathrm{x}) + \mathcal{O}(4) \end{cases}$$

For n = 2 we have the so-called Post-Newtonian Mechanics

By using the Bianchi's identities and introducing two scale lengths m_1, m_2 the field equations in the newtonian limit (n = 1) are

$$\begin{cases} (\triangle - m_2{}^2)R^{(2)}_{tt} + \left[\frac{m_2{}^2}{2} - \frac{m_1{}^2+2m_2{}^2}{6m_1{}^2}\triangle\right]X^{(2)} = -\frac{m_2{}^2\mathcal{X}}{f_X(0)}T^{(0)}_{tt} \\ (\triangle - m_2{}^2)R^{(2)}_{ij} + \left[\frac{m_1{}^2-m_2{}^2}{3m_1{}^2}\partial^2_{ij} - \left(\frac{m_2{}^2}{2} - \frac{m_1{}^2+2m_2{}^2}{6m_1{}^2}\triangle\right)\delta_{ij}\right]X^{(2)} = 0 \\ (\triangle - m_1{}^2)X^{(2)} = \frac{m_1{}^2\mathcal{X}}{f_X(0)}T^{(0)} \end{cases}$$

$$\begin{cases} R_{\alpha\mu\beta\nu;\delta} + R_{\alpha\mu\delta\beta;\nu} + R_{\alpha\mu\nu\delta;\beta} = 0 \\ R_{\alpha\mu\beta\nu}{}^{;\alpha} + R_{\mu\beta;\nu} - R_{\mu\nu;\beta} = 0 \\ 2R_{\alpha\beta}{}^{;\alpha} - R_{;\beta} = 0 \\ 2R_{\alpha\beta}{}^{;\alpha\beta} - \Box R = 0 \end{cases} \qquad \begin{cases} m_1{}^2 \doteq -\frac{f_X(0)}{3f_{XX}(0)+2f_Y(0)+2f_Z(0)} \\ m_2{}^2 \doteq \frac{f_X(0)}{f_Y(0)+4f_Z(0)]} \end{cases}$$

The general solution is found as follows

$$g^{(2)}_{tt}(t,\mathrm{x}) = \frac{1}{2\pi}\int d^3\mathrm{x}'d^3\mathrm{x}''\frac{\mathcal{G}_2(\mathrm{x}',\mathrm{x}'')}{|\mathrm{x}-\mathrm{x}'|}\left[\frac{m_2{}^2\mathcal{X}}{f_X(0)}T^{(0)}_{tt}(t,\mathrm{x}'') - \frac{(m_1{}^2+2m_2{}^2)\mathcal{X}}{6f_X(0)}T^{(0)}t,\mathrm{x}'') + \frac{m_2{}^2-m_1{}^2}{6}X^{(2)}(t,\mathrm{x}'')\right]$$

$$X^{(2)}(t,\mathrm{x}) = \frac{m_1{}^2\mathcal{X}}{f_X(0)}\int d^3\mathrm{x}'\mathcal{G}_1(\mathrm{x},\mathrm{x}')T^{(0)}(t,\mathrm{x}') \qquad g^{(2)}_{ij} = -\frac{\delta_{ij}}{2\pi}\int d^3\mathbf{x}'d^3\mathbf{x}''\frac{\mathcal{G}_2(\mathbf{x}',\mathbf{x}'')}{|\mathbf{x}-\mathbf{x}'|}\left(\frac{m_2{}^2}{2} - \frac{m_1{}^2+2m_2{}^2}{6m_1{}^2}\triangle_{\mathbf{x}''}\right)X^{(2)}(\mathbf{x}'')$$

The "effective Lagrangian" of the FOG in the Newtonian level

By using the Gauss – Bonnet invariant $G_{GB} = X^2 - 4Y + Z$ we have the more simple but in the same time the more general FOG generating the previous solutions. At newtonian level any FOG collapses in an "effective lagrangian".

$$\mathcal{A} = \int d^4x\sqrt{-g}\Big[\mathcal{L} + \mathcal{X}\mathcal{L}_m\Big] = \int d^4x\sqrt{-g}\Big[f(X,Y,Z) + \mathcal{X}\mathcal{L}_m\Big]$$

Identity in the field equation $H^{GB}_{\mu\nu} = H^{X^2}_{\mu\nu} - 4H^Y_{\mu\nu} + H^Z_{\mu\nu} = 0$

$$\mathcal{L} = f(X,Y,Z) = R - \frac{1}{3}\left[\frac{1}{2\mu_1{}^2} + \frac{1}{\mu_2{}^2}\right]R^2 + \frac{R_{\alpha\beta}R^{\alpha\beta}}{\mu_2{}^2}$$

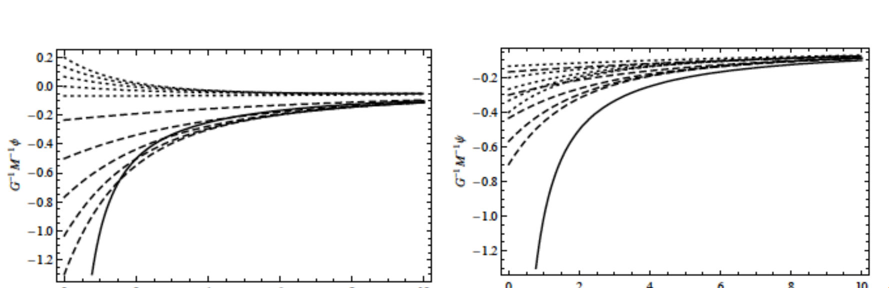
In the case of pointlike source the solutions are

$$ds^2 = (1+2\Phi)dt^2 - (1-2\Psi)\delta_{ij}dx^idx^j$$

$$\begin{cases} X^{(2)} = -\frac{r_g\,m_1{}^2}{f_X(0)}\frac{e^{-m_1|\mathrm{x}|}}{|\mathrm{x}|} \\ \Phi = -\frac{GM}{f_X(0)}\left[\frac{1}{|\mathrm{x}|} + \frac{1}{3}\frac{e^{-m_1|\mathrm{x}|}}{|\mathrm{x}|} - \frac{4}{3}\frac{e^{-m_2|\mathrm{x}|}}{|\mathrm{x}|}\right] \\ \Psi = -\frac{GM}{f_X(0)}\left[\frac{1}{|\mathrm{x}|} - \frac{1}{3}\frac{e^{-m_1|\mathrm{x}|}}{|\mathrm{x}|} - \frac{2}{3}\frac{e^{-m_2|\mathrm{x}|}}{|\mathrm{x}|}\right] \end{cases}$$

For a right physical interpretation of potential we must have

$$f_{XX}(0) + f_Y(0) + 2f_Z(0) < 0$$



The BIRKHOFF THEOREM IN f(R)-Gravity ... but also in FOG

At third order the field equation states the relation between the "*rr*" component and the Ricci scalar $f_1 g^{(2)}_{rr,t} + 2f_2 rR^{(2)}_{,tr} = 0$

In fact, generally, if the "*rr*" component is time-independent and the "*tt*" component is the product of two functions (one of time and other one of the space) the Ricci scalar is time independent. Indeed exists every a redefinition of time gives back a metric time-independent.

But is not verified the Birkhoff theorem in f(R)? ... *It works only at Newtonian level*.

Therefore, the Birkhoff theorem is not a general result for FOG but, on the other hand, in the limit of small velocities and weak fields (which is enough to deal with the Solar System gravitational experiments), one can assume that the gravitational potential is effectively time-independent.

The Noether Symmetry approach to *f(R)*-Gravity

We worked out an approach to obtain time-independent spherically symmetric solutions in *f*-gravity. In order to develop such an approach, we need to deduce a point-like Lagrangian from the general action. Such a Lagrangian can be obtained by imposing the spherical symmetry in the field action. As a consequence, the infinite number of degrees of freedom of the original field theory will be reduced to a finite number.

The Euler-Lagrange equations are in terms of the function *A, B, M, R*. The field equation for *R* corresponds to the constraint among the configuration coordinates.

It is worth noting that the Hessian determinant is zero! The point-like Lagrangian does not depend on *B*. In other words, *B* does not contribute to dynamics and its equation has to be considered as a further constraint equation.

The field-equations approach and the point-like Lagrangian approach differ since the symmetry, in our case the spherical one, can be imposed whether in the field equations, after standard variation with respect to the metric, or directly into the Lagrangian, which becomes point-like.

$$ds^2 = A(r)dt^2 - B(r)dr^2 - M(r)d\Omega$$

$$\mathcal{A} = \int dr\mathcal{L}(A,A',B,B',M,M',R,R')$$

$$\mathcal{L} = -\frac{A^{1/2}f_R}{2MB^{1/2}}M'^2 - \frac{f_R}{A^{1/2}B^{1/2}}A'M' - \frac{Mf_{RR}}{A^{1/2}B^{1/2}}A'R' - \frac{2A^{1/2}f_{RR}}{B^{1/2}}R'M' - A^{1/2}B^{1/2}[(2+MR)f_R - Mf]$$

Field equations approach		Point-like Lagrangian approach
$\downarrow$		$\downarrow$
$\delta\int d^4x\sqrt{-g}f = 0$	$\leftrightarrows$	$\delta\int dr\mathcal{L} = 0$
$\downarrow$		$\downarrow$
$H_{\mu\nu} = \partial_\rho\frac{\partial(\sqrt{-g}f)}{\partial_\rho g^{\mu\nu}} - \frac{\partial(\sqrt{-g}f)}{\partial g^{\mu\nu}} = 0$		$\frac{d}{dr}\nabla_{q'}\mathcal{L} - \nabla_q\mathcal{L} = 0$
	$\leftrightarrows$	
$H = g^{\alpha\beta}H_{\alpha\beta} = 0$		$E_\mathcal{L} = \underline{q}'\cdot\nabla_{q'}\mathcal{L} - \mathcal{L}$
$\downarrow$		$\downarrow$
$H_{tt} = 0$	$\leftrightarrows$	$\frac{d}{dr}\frac{\partial\mathcal{L}}{\partial A'} - \frac{\partial\mathcal{L}}{\partial A} = 0$
$H_{rr} = 0$	$\leftrightarrows$	$\frac{d}{dr}\frac{\partial\mathcal{L}}{\partial B'} - \frac{\partial\mathcal{L}}{\partial B} \propto E_\mathcal{L} = 0$
$H_{\theta\theta} = \csc^2\theta H_{\phi\phi} = 0$	$\leftrightarrows$	$\frac{d}{dr}\frac{\partial\mathcal{L}}{\partial M'} - \frac{\partial\mathcal{L}}{\partial M} = 0$
$H = A^{-1}H_{tt} - B^{-1}H_{rr} - 2M^{-1}\csc^2\theta H_{\phi\phi} = 0$	$\leftrightarrows$	A combination of the above equations

We find a relation between the metric potentials

$$B = \frac{2M^2f_{RR}A'R' + 2Mf_RA'M' + 4AMf_{RR}M'R' + Af_RM'^2}{2AM[(2+MR)f_R - Mf]}$$

If *f(R)* = *R* we find the well known condition *A B* = *1*. By using the previous constraint condition we obtain a dynamics with tree generalized coordinates and not four. Than the point-like lagrangian becomes

$$\mathbf{L} = \underline{q}'^t\,\hat{\mathbf{L}}\,\underline{q}' = \frac{[(2+MR)f_R - fM]}{M}[2M^2f_{RR}A'R' + 2MM'(f_RA' + 2Af_{RR}R') + Af_RM'^2]$$

By applying the Noether Symmetry Approach we find a constant of motion for a power law *f(R)* = R^s.

$$\Sigma_0 = \underline{\alpha}\cdot\nabla_{q'}\mathbf{L} = 2skMR^{2s-3}[2s + (s-1)MR][(s-2)RA' - (2s^2-3s+1)AR']$$

The solution, mathematically exact, for s = 5/2, is

$$ds^2 = \frac{1}{\sqrt{5}}(k_2 + k_1r)dt^2 - \frac{1}{2}\left(\frac{1}{1+\frac{k_2}{k_1r}}\right)dr^2 - r^2d\Omega$$

The Galactic Rotation Curves

Generally the superposition principle is not valid in FOG like in GR, but we are at newtonian level (linear field equations). Then we can assume the gravitational potential and the galactic curve rotation, respectively, are

$$\Phi(\mathbf{x}) = -G\int d^3\mathbf{x}'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\left[1 + \frac{1}{3}e^{-\mu_1|\mathbf{x}-\mathbf{x}'|} - \frac{4}{3}e^{-\mu_2|\mathbf{x}-\mathbf{x}'|}\right] \qquad \frac{v_c(|\mathbf{x}|)^2}{|\mathbf{x}|} = \frac{\partial\Phi(\mathbf{x})}{\partial|\mathbf{x}|}$$

There is a fundamental difference amoung the GR and FOG

validity or not of Gauss theorem

GR case

$$v_c(r)^2 = \frac{GM(r)}{r} = \frac{4\pi G}{r}\int_0^r dy\,y^2\rho(y)$$

FOG case

Only a numerical integration …

The galactic structure is a rotating system but the newtonian limit of the axially symmetry is the spherical one. Then the Galaxy is still.

Generally the potential is the sum of three components: bulge, disk and dark matter.

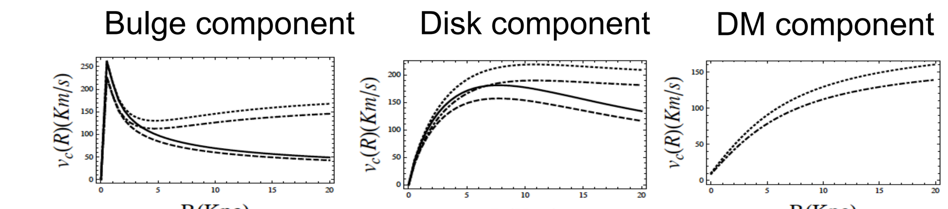
Galaxy	M_b	ξ_b	γ	M_d	ξ_d	M_{DM}	ξ_{DM}	α	Ξ
Milky Way	0.77	0.5	1.5	5.20	3.5	1.68	5.5	0.50	20
NGC 3198	0	/	/	2.60	3.5	0.84	5.5	0.53	20

The unity of mass is 10^{10} solar mass; The unity of distance is 1 Kpc.

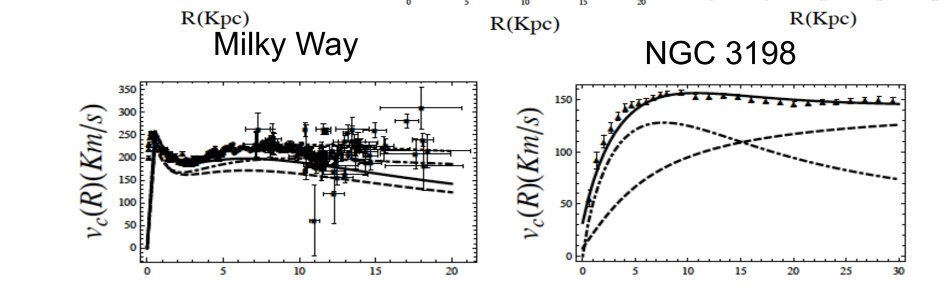
$$\begin{cases} \rho_{bulge}(r) = \frac{M_b}{2\pi\xi_b{}^{3-\gamma}\Gamma(\frac{3-\gamma}{2})}\frac{e^{-\frac{r^2}{\xi_b{}^2}}}{r^\gamma} \\ \sigma_{disk}(R) = \frac{M_d}{2\pi\xi_d{}^2}e^{-\frac{R}{\xi_d}} \\ \rho_{DM}(r) = \frac{\alpha M_{DM}}{\pi(4-\pi)\xi_{DM}{}^3}\frac{1}{1+\frac{r^2}{\xi_{DM}{}^2}} \end{cases}$$

GR: Dashed line; GR + DM: Dashed and dotted line; FOG: Solid line; FOG + DM: Dotted line. $\mu_1 = 10^{-2}$ Kpc^{-1}, $\mu_2 = 10^2$ Kpc^{-1}

Bulge component   Disk component   DM component



Milky Way   NGC 3198



We need DM component!

Gravitational Lensing by Extended Matter Source

The starting point $ds^2 = (1+2\Phi)dt^2 - (1-2\Psi)\delta_{ij}dx^idx^j$

The metric potentials

$$\begin{cases} \Phi = -G\int d^3\mathbf{x}'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\left[1 + \frac{1}{3}e^{-\mu_1|\mathbf{x}-\mathbf{x}'|} - \frac{4}{3}e^{-\mu_2|\mathbf{x}-\mathbf{x}'|}\right] \\ \Psi = -G\int d^3\mathbf{x}'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\left[1 - \frac{1}{3}e^{-\mu_1|\mathbf{x}-\mathbf{x}'|} - \frac{2}{3}e^{-\mu_2|\mathbf{x}-\mathbf{x}'|}\right] \end{cases}$$

The geodesic motion $\dot{\mathbf{u}} = -2\left[\nabla_\perp\Psi + \frac{1}{2}\nabla(\Phi - \Psi)\right]$

In GR the metric potentials are equal, but in FOG we have

$$\triangle(\Phi - \Psi) = \frac{m_1{}^2 - m_2{}^2}{3m_1{}^2}\int d^3\mathbf{x}'\mathcal{G}_2(\mathbf{x},\mathbf{x}')\,\triangle_{\mathbf{x}'}X^{(2)}(\mathbf{x}')$$

The deflection angle is given by $\vec{\alpha} = -\int_{\lambda_i}^{\lambda_f}\frac{d\mathbf{u}}{d\lambda}d\lambda = \int_{z_i}^{z_f}\left[\nabla_{\vec{\xi}}(\Phi + \Psi) + \hat{z}\,\partial_z(\Phi - \Psi)\right]dz$

which in the limit of planar and point-like source becomes

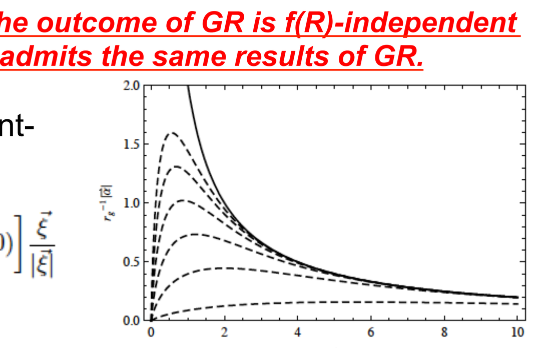
$$\vec{\alpha} = 4G\int d^2\vec{\xi}'\Sigma(\vec{\xi}')\left[\frac{1}{|\vec{\xi}-\vec{\xi}'|} - |\vec{\xi}-\vec{\xi}'|\,\mathcal{F}_{\mu_2}(\vec{\xi},\vec{\xi}')\right]\frac{\vec{\xi}-\vec{\xi}'}{|\vec{\xi}-\vec{\xi}'|}$$

$$\mathcal{F}_{\mu_2}(\vec{\xi},\vec{\xi}') = \int_0^\infty dz\frac{(1 + \mu_2\Delta(\vec{\xi},\vec{\xi}',z,0))}{\Delta(\vec{\xi},\vec{\xi}',z,0)^3}e^{-\mu_2\Delta(\vec{\xi},\vec{\xi}',z,0)}$$

The correction to the outcome of GR is f(R)-independent f(R)-Gravity admits the same results of GR.

In the limit of point-like source

$$\vec{\alpha} = 2r_g\left[\frac{1}{|\vec{\xi}|} - |\vec{\xi}|\,\mathcal{F}_{\mu_2}(\vec{\xi},0)\right]\frac{\vec{\xi}}{|\vec{\xi}|}$$



Correction to the Einstein ring $\theta = \pm\theta_E\left[1 - \frac{\theta_E{}^2}{2}\mathcal{F}(\theta_E)\right]$