

The Scalar Tensor Fourth Order Gravity: solutions, astrophysical applications and conformal transformations

Arturo Stabile

arturo.stabile@gmail.com

in collaboration with S. Capozziello, G. Scelza, An. Stabile

Department of Engineering
University of Sannio
Benevento - Italy

Vietri sul mare, 16 April 2014

Abstract

The first step of comparison between the General Relativity and the so-called Scalar Tensor Fourth Order Gravity must be performed in the same framework of astrophysical applications that demonstrated the validity of Einsteins theory with respect to the Newtonian mechanics.

By considering other curvature invariants and also the presence of non-minimally coupled scalar field the solutions for the spherically symmetric systems are shown in the limit of weak field.

The solutions are applied in the astrophysical framework: the galactic rotation curves and the gravitational lensing.

Finally the mathematical relations between the Jordan and Einstein frames in the weak field limit are shown.

Table of contents

- Scalar Tensor Fourth Order Gravity

A. Stabile

STFOG

Frameworks

Newtonian
limit

Rotation curve

Galactic
rotation curve

Gravitational
lensing

Weak
conformal
transformation

Conclusions

References



Stabile A., Stabile An., Capozziello S., Physical Review D **88**, 124011 (2013)



Stabile A., Capozziello S., Phys. Rev. D **87**, 064002 (2013)



Stabile A., Stabile An., Phys. Rev. D **85**, 044014 (2012)



Stabile A., Scelza G., Phys. Rev. D. **84**, 124023 (2011)



Stabile A., Phys. Rev. D **82**, 064021 (2010)



Stabile A., Phys. Rev. D **82**, 124026 (2010)



Capozziello S., Stabile A., Class. Quant. Grav. **26**, 085019 (2009)

Table of contents

Scalar Tensor
Fourth Order
Gravity

A. Stabile

STFOG

Frameworks

Newtonian
limit

Rotation curve

Galactic
rotation curve

Gravitational
lensing

Weak
conformal
transformation

Conclusions

- Scalar Tensor Fourth Order Gravity
- Frameworks for the analysis and for physical applications

References



Stabile A., Stabile An., Capozziello S., Physical Review D **88**, 124011 (2013)



Stabile A., Capozziello S., Phys. Rev. D **87**, 064002 (2013)



Stabile A., Stabile An., Phys. Rev. D **85**, 044014 (2012)



Stabile A., Scelza G., Phys. Rev. D. **84**, 124023 (2011)



Stabile A., Phys. Rev. D **82**, 064021 (2010)



Stabile A., Phys. Rev. D **82**, 124026 (2010)



Capozziello S., Stabile A., Class. Quant. Grav. **26**, 085019 (2009)

Table of contents

Scalar Tensor
Fourth Order
Gravity

A. Stabile

STFOG

Frameworks

Newtonian
limit

Rotation curve

Galactic
rotation curve

Gravitational
lensing

Weak
conformal
transformation

Conclusions

- Scalar Tensor Fourth Order Gravity
- Frameworks for the analysis and for physical applications
- The Newtonian limit of the Scalar Tensor Fourth Order Gravity

References



Stabile A., Stabile An., Capozziello S., Physical Review D **88**, 124011 (2013)



Stabile A., Capozziello S., Phys. Rev. D **87**, 064002 (2013)



Stabile A., Stabile An., Phys. Rev. D **85**, 044014 (2012)



Stabile A., Scelza G., Phys. Rev. D. **84**, 124023 (2011)



Stabile A., Phys. Rev. D **82**, 064021 (2010)



Stabile A., Phys. Rev. D **82**, 124026 (2010)



Capozziello S., Stabile A., Class. Quant. Grav. **26**, 085019 (2009)

- Scalar Tensor Fourth Order Gravity
- Frameworks for the analysis and for physical applications
- The Newtonian limit of the Scalar Tensor Fourth Order Gravity
- Galactic rotation curves: theoretical predictions and experimental evidence

References







-  Stabile A., Stabile An., Capozziello S., Physical Review D **88**, 124011 (2013)
-  Stabile A., Capozziello S., Phys. Rev. D **87**, 064002 (2013)
-  Stabile A., Stabile An., Phys. Rev. D **85**, 044014 (2012)
-  Stabile A., Scelza G., Phys. Rev. D. **84**, 124023 (2011)
-  Stabile A., Phys. Rev. D **82**, 064021 (2010)
-  Stabile A., Phys. Rev. D **82**, 124026 (2010)
-  Capozziello S., Stabile A., Class. Quant. Grav. **26**, 085019 (2009)

Table of contents

Scalar Tensor
Fourth Order
Gravity

A. Stabile

STFOG

Frameworks

Newtonian
limit

Rotation curve

Galactic
rotation curve

Gravitational
lensing

Weak
conformal
transformation

Conclusions

- Scalar Tensor Fourth Order Gravity
- Frameworks for the analysis and for physical applications
- The Newtonian limit of the Scalar Tensor Fourth Order Gravity
- Galactic rotation curves: theoretical predictions and experimental evidence
- Gravitational lensing: theoretical predictions

References



Stabile A., Stabile An., Capozziello S., *Physical Review D* **88**, 124011 (2013)



Stabile A., Capozziello S., *Phys. Rev. D* **87**, 064002 (2013)



Stabile A., Stabile An., *Phys. Rev. D* **85**, 044014 (2012)



Stabile A., Scelza G., *Phys. Rev. D* **84**, 124023 (2011)



Stabile A., *Phys. Rev. D* **82**, 064021 (2010)



Stabile A., *Phys. Rev. D* **82**, 124026 (2010)



Capozziello S., Stabile A., *Class. Quant. Grav.* **26**, 085019 (2009)

- Scalar Tensor Fourth Order Gravity
- Frameworks for the analysis and for physical applications
- The Newtonian limit of the Scalar Tensor Fourth Order Gravity
- Galactic rotation curves: theoretical predictions and experimental evidence
- Gravitational lensing: theoretical predictions
- Weak Conformal transformations

References



Stabile A., Stabile An., Capozziello S., Physical Review D **88**, 124011 (2013)



Stabile A., Capozziello S., Phys. Rev. D **87**, 064002 (2013)



Stabile A., Stabile An., Phys. Rev. D **85**, 044014 (2012)



Stabile A., Scelza G., Phys. Rev. D. **84**, 124023 (2011)



Stabile A., Phys. Rev. D **82**, 064021 (2010)










Stabile A., Phys. Rev. D **82**, 124026 (2010)



Capozziello S., Stabile A., Class. Quant. Grav. **26**, 085019 (2009)

- Scalar Tensor Fourth Order Gravity
- Frameworks for the analysis and for physical applications
- The Newtonian limit of the Scalar Tensor Fourth Order Gravity
- Galactic rotation curves: theoretical predictions and experimental evidence
- Gravitational lensing: theoretical predictions
- Weak Conformal transformations
- Conclusions

References

-  [Stabile A., Stabile An., Capozziello S., Physical Review D **88**, 124011 \(2013\)](#)
-  [Stabile A., Capozziello S., Phys. Rev. D **87**, 064002 \(2013\)](#)
-  [Stabile A., Stabile An., Phys. Rev. D **85**, 044014 \(2012\)](#)
-  [Stabile A., Scelza G., Phys. Rev. D. **84**, 124023 \(2011\)](#)
-  [Stabile A., Phys. Rev. D **82**, 064021 \(2010\)](#)
-  [Stabile A., Phys. Rev. D **82**, 124026 \(2010\)](#)
-  [Capozziello S., Stabile A., Class. Quant. Grav. **26**, 085019 \(2009\)](#)

Scalar Tensor Fourth Order Gravity (STFOG)

The action

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[f(R, R_{\alpha\beta} R^{\alpha\beta}, \phi) + \omega(\phi) \phi_{;\alpha} \phi^{;\alpha} + \mathcal{X} \mathcal{L}_{matter} \right]$$

- $R \rightarrow$ Ricci scalar
- $R_{\alpha\beta} R^{\alpha\beta} \doteq Y \rightarrow$ Ricci tensor square
- $\phi \rightarrow$ scalar field
- $\mathcal{L}_{matter} \rightarrow$ minimally coupled ordinary matter
- $g \rightarrow$ determinant of metric tensor $g_{\mu\nu}$
- $\mathcal{X} = 8\pi G/c^2$ but we use $c = 1$
- $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$, $R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} + \dots \rightarrow$ convention
- $2\Gamma^\mu_{\alpha\beta} = g^{\mu\sigma} (g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma})$
- $(+ - - -) \rightarrow$ the adopted signature

Scalar Tensor Fourth Order Gravity

The field equations by applying $\delta(\cdot) \rightarrow \delta g_{\mu\nu} \frac{\delta(\cdot)}{\delta g_{\mu\nu}} + \delta\phi \frac{\delta(\cdot)}{\delta\phi}$

$$f_R R_{\mu\nu} - \frac{f + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}}{2} g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu}\square f_R + 2f_Y R_{\mu}{}^{\alpha} R_{\alpha\nu} - 2[f_Y R^{\alpha}{}_{(\mu};\nu)\alpha] \\ + \square[f_Y R_{\mu\nu}] + [f_Y R_{\alpha\beta}]^{;\alpha\beta} g_{\mu\nu} + \omega(\phi)\phi_{;\mu}\phi_{;\nu} = \mathcal{X} T_{\mu\nu}$$

$$2\omega(\phi)\square\phi + \omega_{\phi}(\phi)\phi_{;\alpha}\phi^{;\alpha} - f_{\phi} = 0$$

$$f_R R + 2f_Y R_{\alpha\beta} R^{\alpha\beta} - 2f + \square[3f_R + f_Y R] + 2[f_Y R^{\alpha\beta}]_{;\alpha\beta} - \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} = \mathcal{X} T$$

- $T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \rightarrow$ the tensor of matter
- $T = T^{\rho}{}_{\rho} \rightarrow$ trace of matter tensor
- $f_R = \frac{df}{dR}$, $f_Y = \frac{df}{dY}$, $\omega_{\phi} = \frac{d\omega}{d\phi}$, $f_{\phi} = \frac{df}{d\phi}$, $\square = ;\sigma^{;\sigma}$

The scalar field ϕ is coupled with the geometry R , $R_{\mu\nu}$ (as the classical **scalar tensor gravity**) and the field equations are differential equations of **fourth order**.

The technically possible frameworks of STFOG

The first two interesting frameworks are

Newtonian and Post-Newtonian limit	Weak field limit
$g_{\mu\nu} \sim \begin{pmatrix} 1 + g_{tt}^{(2)}(t, \mathbf{x}) + g_{tt}^{(4)}(t, \mathbf{x}) & g_{ti}^{(3)}(t, \mathbf{x}) \\ g_{ti}^{(3)}(t, \mathbf{x}) & -\delta_{ij} + g_{ij}^{(2)}(t, \mathbf{x}) \end{pmatrix}$	$g_{\mu\nu} \sim \eta_{\mu\nu} + h_{\mu\nu}(x)$
$\phi \sim \phi^{(0)} + \phi^{(2)}(t, \mathbf{x})$	$\phi \sim \phi^{(0)} + \phi^{(1)}(x)$
$;_{\mu} \sim ,_{\mu} + \tilde{;}_{\mu} = -\nabla + \tilde{\partial}_{\mu} \quad \square \sim -\Delta + \tilde{\square}$	$;_{\mu} \sim ,_{\mu} = \partial_{\mu} \quad \square \sim \square_{\eta}$
Slow motion in the spherically symmetric systems	Gravitational waves

- $x = (t, \mathbf{x})$
- The approximation level is driven by an expansion of the powers of $1/c$
- The approximation level is driven by the linearization of field equations $\mathcal{O}(h_{\mu\nu})^2 \ll 1$
- In the Newtonian limit the space time is time independent

The Newtonian limit of STFOG

By setting $f_R(0, 0, \phi^{(0)}) = 1$, $\omega(\phi^{(0)}) = 1/2$ the field equations are

$$(\Delta - m_Y^2)\Delta\Phi + \left[\frac{m_Y^2}{2} - \frac{m_R^2 + 2m_Y^2}{6m_R^2} \Delta \right] R + m_Y^2 f_{R\phi}(0, 0, \phi^{(0)}) \Delta\varphi = -m_Y^2 \mathcal{X} T_{tt}$$

$$(\Delta - m_R^2)R - 3m_R^2 f_{R\phi}(0, 0, \phi^{(0)}) \Delta\varphi = m_R^2 \mathcal{X} T$$

$$(\Delta - m_\phi^2)\varphi + f_{R\phi}(0, 0, \phi^{(0)}) R = 0$$

where

$$g_{\mu\nu} \sim \begin{pmatrix} 1 + 2\Phi(\mathbf{x}) & 0 \\ 0 & -\delta_{ij} \end{pmatrix}$$

The derivatives are calculated on the Minkowskian background

$$m_R^2 \doteq - \frac{1}{3f_{RR}(0,0,\phi^{(0)}) + 2f_Y(0,0,\phi^{(0)})}$$

$$m_Y^2 \doteq \frac{1}{f_Y(0,0,\phi^{(0)})}$$

$$m_\phi^2 \doteq -f_{\phi\phi}(0, 0, \phi^{(0)})$$

- $T_{tt} = T = \rho(\mathbf{x}) \rightarrow$ the energy mass density of the source
- $T = T^\rho{}_\rho \rightarrow$ trace of matter tensor
- The theory is parameterized by the coefficients of Taylor expansion of Lagrangian.

The Newtonian limit of STFOG: the pointlike gravitational potential

The scalar field φ and also the Ricci scalar R are auxiliary fields. The physical outcome is

$$\Phi(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|} \left[1 + g(\xi, \eta) e^{-m_R \tilde{k}_R |\mathbf{x}|} + [1/3 - g(\xi, \eta)] e^{-m_R \tilde{k}_\phi |\mathbf{x}|} - \frac{4}{3} e^{-m_Y |\mathbf{x}|} \right]$$

$$\text{where } g(\xi, \eta) = \frac{1 - \eta^2 + \xi + \sqrt{\eta^4 + (\xi - 1)^2 - 2\eta^2(\xi + 1)}}{6\sqrt{\eta^4 + (\xi - 1)^2 - 2\eta^2(\xi + 1)}}, \quad \tilde{k}_{R, \phi}^2 = \frac{1 - \xi + \eta^2 \pm \sqrt{(1 - \xi + \eta^2)^2 - 4\eta^2}}{2}, \quad \eta = \frac{m_\phi}{m_R}$$

and $\xi = 3f_{R\phi}(0, 0, \phi^{(0)})^2$.

The effective Lagrangian of STFOG is given as

$$R + \frac{f_{RR}(0, 0, \phi^{(0)})}{2} R^2 + \frac{f_{\phi\phi}(0, 0, \phi^{(0)})}{2} \varphi^2 + f_{R\phi}(0, 0, \phi^{(0)}) R \varphi + f_Y(0, 0, \phi^{(0)}) R_{\alpha\beta} R^{\alpha\beta} + \frac{|\nabla\varphi|^2}{2}$$

... and if we add the terms proportional to $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ and/or to $\square^k R$? **The outcome is the same!!**

- The Gauss-Bonnet invariant ...
- \square^k is the four divergence ...

The Newtonian limit of STFOG: Classes of the gravitational potentials (A, B)

Case	Gravitational potential	Free parameters
A	$-\frac{GM}{ x } \left[1 + \frac{1}{3} e^{-m_R x } \right]$	$m_R^2 = -\frac{1}{3f_{RR}(0)}$
B	$-\frac{GM}{ x } \left[1 + \frac{1}{3} e^{-m_R x } - \frac{4}{3} e^{-m_Y x } \right]$	$m_R^2 = -\frac{1}{3f_{RR}(0,0) + 2f_Y(0,0)}$ $m_Y^2 = \frac{1}{f_Y(0,0)}$

- case A $\rightarrow f(R)$
- case B $\rightarrow f(R, R_{\alpha\beta} R^{\alpha\beta})$
- case C $\rightarrow f(R, \phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$
- case D $\rightarrow f(R, R_{\alpha\beta} R^{\alpha\beta}, \phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$

The Newtonian limit of STFOG: Classes of the gravitational potentials (C)

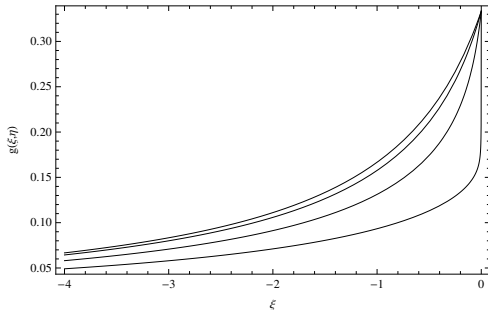
Gravitational potential	Free parameters
$-\frac{GM}{ x } \left[1 + g(\xi, \eta) e^{-m_R \tilde{k}_R x } + [1/3 - g(\xi, \eta)] e^{-m_R \tilde{k}_\phi x } \right]$	$m_R^2 = -\frac{1}{3f_{RR}(0,1)}$ $m_\phi^2 = -\frac{f_{\phi\phi}(0,1)}{2\omega(1)}$ $\xi = \frac{3f_{R\phi}(0,1)^2}{2\omega(1)}$ $\eta = \frac{m_\phi}{m_R}$ $g(\xi, \eta) = \frac{1 - \eta^2 + \xi + \sqrt{\eta^4 + (\xi - 1)^2 - 2\eta^2(\xi + 1)}}{6\sqrt{\eta^4 + (\xi - 1)^2 - 2\eta^2(\xi + 1)}}$ $\tilde{k}_{R,\phi}^2 = \frac{1 - \xi + \eta^2 \pm \sqrt{(1 - \xi + \eta^2)^2 - 4\eta^2}}{2}$

The Newtonian limit of STFOG: Classes of the gravitational potentials (D)

Gravitational potential	Free parameters
$-\frac{GM}{ \mathbf{x} } \left[1 + g(\xi, \eta) e^{-m_R \tilde{k}_R \mathbf{x} } + \right. \\ \left. + [1/3 - g(\xi, \eta)] e^{-m_R \tilde{k}_\phi \mathbf{x} } - \frac{4}{3} e^{-m_Y \mathbf{x} } \right]$	$m_R^2 = -\frac{1}{3f_{RR}(0,0,1) + 2f_Y(0,0,1)}$
	$m_Y^2 = \frac{1}{f_Y(0,0,1)}$
	$m_\phi^2 = -\frac{f_\phi \phi^{(0,0,1)}}{2\omega(1)}$
	$\xi = \frac{3f_R \phi^{(0,0,1)^2}}{2\omega(1)}$
	$\eta = \frac{m_\phi}{m_R}$
	$g(\xi, \eta) = \\ = \frac{1 - \eta^2 + \xi + \sqrt{\eta^4 + (\xi - 1)^2 - 2\eta^2(\xi + 1)}}{6\sqrt{\eta^4 + (\xi - 1)^2 - 2\eta^2(\xi + 1)}} \\ \tilde{k}_{R,\phi}^2 = \frac{1 - \xi + \eta^2 \pm \sqrt{(1 - \xi + \eta^2)^2 - 4\eta^2}}{2}$

The Newtonian limit of STFOG: The crucial choice of the coefficient and of the theory!

The potential does not depend only on the masses propagating (cases A, B) but also on the function modeling the Yukawa corrections (case C, D).



Plot of coefficient $g(\xi, \eta)$ (case C) with respect to quantity ξ for $0 \leq \eta \leq 0.99$ with step 0.33.

$$g(\xi, \eta) = \frac{1 - \eta^2 + \xi + \sqrt{\eta^4 + (\xi - 1)^2 - 2\eta^2(\xi + 1)}}{6\sqrt{\eta^4 + (\xi - 1)^2 - 2\eta^2(\xi + 1)}}, \quad \tilde{k}_{R, \phi}^2 = \frac{1 - \xi + \eta^2 \pm \sqrt{(1 - \xi + \eta^2)^2 - 4\eta^2}}{2}, \quad \eta = \frac{m_\phi}{m_R} \text{ and}$$

$$\xi = \frac{3f_{R\phi}(0, 0, \phi^{(0)})^2}{2\omega(\phi^{(0)})}.$$

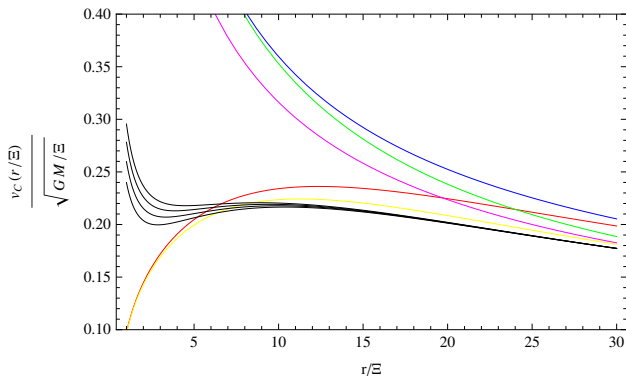
The rotation curve in STFOG

- **In STFOG the Gauss theorem is not verified!**
- The case of source ball-like is different from the pointlike one: the potential depends on the dimension of the source.
- Generally for any term $\propto \frac{e^{-mr}}{r}$ we have a geometric factor multiplying the Yukawa term.
- In the case of a ball with radius Ξ we find $F(m\Xi) = 3 \frac{m\Xi \cosh m\Xi - \sinh m\Xi}{m^3 \Xi^3}$

$$\Phi_{ball}(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|} \left[1 + g(\xi, \eta) F(m_R \tilde{k}_R \Xi) e^{-m_R \tilde{k}_R |\mathbf{x}|} + [1/3 - g(\xi, \eta)] F(m_R \tilde{k}_\phi \Xi) e^{-m_R \tilde{k}_\phi |\mathbf{x}|} - \frac{4}{3} F(m_Y \Xi) e^{-m_Y |\mathbf{x}|} \right]$$

- The rotation curve of the body around the ball source is easily obtained $v_c(r) = \sqrt{r \frac{\partial \Phi_{ball}(r)}{\partial r}}$ where $|\mathbf{x}| = r$.
- A famous and satisfactory potential for the galactic rotation curves is the Sanders potential $\Phi_{SP}(r) = -\frac{GM}{r} \left[1 + \alpha e^{-m_S r} \right]$ where α and m_S are free parameters.

The rotation curve in STFOG and in General Relativity



Plot circular velocity by assuming for a ball source with mass M and radius Ξ . The behaviours are the following: red line (case D), blue line (case C), yellow line (case B), green line (case A), magenta line (case of GR). The black lines are the Sanders model for $-0.95 < \alpha < -0.92$. The values of free parameters are: $\xi = -5$, $\eta = .3$, $m_\gamma = 1.5 * m_R$, $m_S = 1.5 * m_R$, $m_R = .1 * \Xi^{-1}$

The rotation curve in FOG: The galactic rotation

- The model of the galaxy: bulge, disk and (why not) dark matter component
- The gauss theorem is out but the field equations are linear → Superposition principle

The model for the matter distribution in the galaxy

$$\left\{ \begin{array}{l} \rho_{bulge}(r) = \frac{M_b}{2\pi \xi_b^{3-\gamma} \Gamma(\frac{3-\gamma}{2})} e^{-\frac{r^2}{\xi_b^2}} \frac{1}{r^\gamma} \\ \sigma_{disk}(R) = \frac{M_d}{2\pi \xi_d^2} e^{-\frac{R}{\xi_d}} \\ \rho_{DM}(r) = \frac{\alpha M_{DM}}{\pi (4-\pi) \xi_{DM}^3} \frac{1}{1 + \frac{r^2}{\xi_{DM}^2}} \end{array} \right.$$

The galactic potential is performed by numerical integration

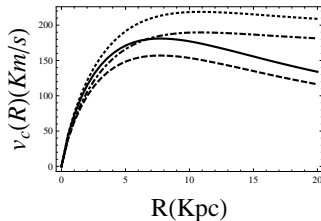
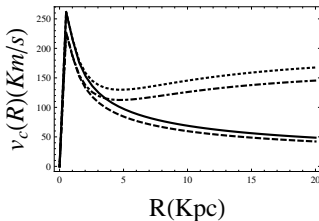
$$\Phi(\mathbf{x}) = \Phi(r, R, z) = -G \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \left[1 + \frac{1}{3} e^{-m_R |\mathbf{x} - \mathbf{x}'|} - \frac{4}{3} e^{-m_Y |\mathbf{x} - \mathbf{x}'|} \right]$$

The rotation curve in the galactic plane → $v_c(R) = \sqrt{R \frac{\partial}{\partial R} \Phi(R, R, 0)}$

The rotation curve in FOG: The galactic rotation curve

The rotation curve induced by

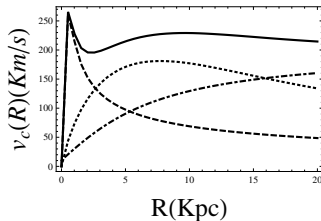
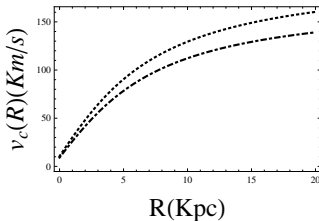
- bulge component: GR (dashed line), GR + DM (dashed and dotted line), FOG (solid line) and FOG + DM (dotted line).
- disk component: GR (dashed line), GR + DM (dashed and dotted line), FOG (solid line) and FOG + DM (dotted line)



The rotation curve in FOG: The galactic rotation curve

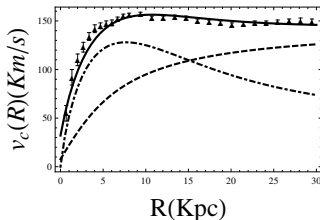
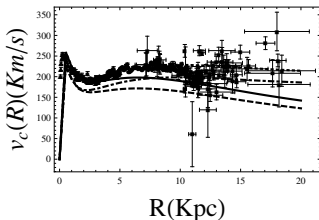
The rotation curve induced by

- DM component: GR + DM (dashed and dotted line), FOG + DM (dotted line).
- galactic sturcture: GR (dashed line), GR + DM (dashed and dotted line), FOG (solid line) and FOG + DM (dotted line): GR (dashed line), GR + DM (dashed and dotted line), FOG (solid line) and FOG + DM (dotted line)



The experimental galactic rotation curves vs FOG

Superposition of theoretical behaviors (GR (dashed line), GR + DM (dashed and dotted line), FOG (solid line), FOG + DM (dotted line)) on the experimental data for Milky Way and NGC 3198



Galaxy	M_b	ξ_b	γ	M_d	ξ_d	M_{DM}	ξ_{DM}	α	Ξ
Milky Way	0.77	0.5	1.5	5.20	3.5	1.68	5.5	0.50	20
NGC 3198	0	/	/	2.60	3.5	0.84	5.5	0.53	20

The units $\rightarrow (10^{10} M_{\odot}, 1\text{Kpc})$. The data fitting $\rightarrow m_R = 10^{-2}\text{Kpc}^{-1}$, $m_Y = 10^2\text{Kpc}^{-1}$

We need Dark Matter component

The gravitational lensing by extended matter distribution in FOG

The deflection angle of photon, by solving the geodesic motion, is given as

$$\vec{\alpha} = \int_{z_i}^{z_f} \left[\nabla_{\vec{\xi}} (\Phi + \Psi) + \hat{z} \partial_z (\Phi - \Psi) \right] dz$$

where $g_{ij}^{(2)} = \Psi \delta_{ij}$. In presence of the generic matter distribution we find

$$\begin{cases} \Phi + \Psi = -2G \int d^2 \vec{\xi}' dz' \frac{\rho(\vec{\xi}', z')}{\Delta(\vec{\xi}, \vec{\xi}', z, z')} + 2G \int d^2 \vec{\xi}' dz' \frac{\rho(\vec{\xi}', z')}{\Delta(\vec{\xi}, \vec{\xi}', z, z')} e^{-\mu_Y \Delta(\vec{\xi}, \vec{\xi}', z, z')} \\ \Phi - \Psi = -\frac{2G}{3} \int d^2 \vec{\xi}' dz' \frac{\rho(\vec{\xi}', z')}{\Delta(\vec{\xi}, \vec{\xi}', z, z')} \left[e^{-m_R \Delta(\vec{\xi}, \vec{\xi}', z, z')} - e^{-m_Y \Delta(\vec{\xi}, \vec{\xi}', z, z')} \right] \end{cases}$$

The gravitational lensing by extended 2D matter distribution in FOG

Then the deflection angle

$$\begin{aligned} \vec{\alpha} = & 2G \int_{z_i}^{z_f} d^2 \vec{\xi}' dz' dz \frac{\rho(\vec{\xi}', z')(\vec{\xi} - \vec{\xi}')}{\Delta(\vec{\xi}, \vec{\xi}', z, z')^3} \\ & - 2G \int_{z_i}^{z_f} d^2 \vec{\xi}' dz' dz \frac{\rho(\vec{\xi}', z')[1 + m_Y \Delta(\vec{\xi}, \vec{\xi}', z, z')]}{\Delta(\vec{\xi}, \vec{\xi}', z, z')^3} e^{-m_Y \Delta(\vec{\xi}, \vec{\xi}', z, z')} (\vec{\xi} - \vec{\xi}') \\ & + \frac{2G}{3} \hat{z} \int_{z_i}^{z_f} d^2 \vec{\xi}' dz' dz \frac{\rho(\vec{\xi}', z')(z - z')}{\Delta(\vec{\xi}, \vec{\xi}', z, z')^3} \left[\left(1 + m_R \Delta(\vec{\xi}, \vec{\xi}', z, z') \right) e^{-m_R \Delta(\vec{\xi}, \vec{\xi}', z, z')} \right. \\ & \left. - \left(1 + m_Y \Delta(\vec{\xi}, \vec{\xi}', z, z') \right) e^{-m_Y \Delta(\vec{\xi}, \vec{\xi}', z, z')} \right] \end{aligned}$$

If the lens belongs to the plane ortogonal to the direction of the photon (hypothesis of the "thin lens")

$f(R)$ -gravity = General Relativity

$$\vec{\alpha} = 4G \int d^2 \vec{\xi}' \Sigma(\vec{\xi}') \left[\frac{1}{|\vec{\xi} - \vec{\xi}'|} - |\vec{\xi} - \vec{\xi}'| \mathcal{F}_{m_Y}(\vec{\xi}, \vec{\xi}') \right] \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|}$$

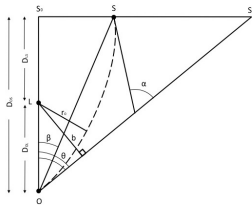
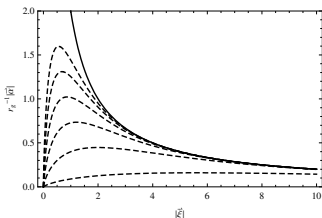
The gravitational lensing by pointlike source in FOG

In the case of pointlike source *i.e.* $\Sigma(\vec{\xi}') = M \delta^{(2)}(\vec{\xi}')$

$$\vec{\alpha} = 2 r_g \left[\frac{1}{|\vec{\xi}|} - |\vec{\xi}| \mathcal{F}_{m_Y}(\vec{\xi}, 0) \right] \frac{\vec{\xi}}{|\vec{\xi}|}$$

and the correction to the images position is given by $\theta = \theta_{\pm}^{GR} \mp \frac{\theta_E^2}{\sqrt{\beta^2 + 4\theta_E^2}} \mathcal{F}(\theta_{\pm}^{GR}) \theta_{\pm}^{GR2}$ where

$$\mathcal{F}(\theta) = \int_0^{\infty} dz \frac{(1+m_Y D_{OL} \sqrt{\theta^2+z^2})}{\sqrt{(\theta^2+z^2)^3}} e^{-m_Y D_{OL} \sqrt{\theta^2+z^2}} \text{ and } \theta_E = \sqrt{\frac{2 r_g D_{LS}}{D_{OL} D_{OS}}} \text{ is the Einstein angle}$$



The Newtonian limit of the Scalar Tensor Gravity in the Jordan frame

- The action of the scalar-tensor theory of gravity in the Jordan frame

$$\mathcal{A}^{JF} = \int d^4x \sqrt{-g} \left[\phi R + V(\phi) + \omega(\phi) \phi_{;\alpha} \phi^{;\alpha} + \mathcal{X} \mathcal{L}_m \right]$$

- The field equations for the fields $g_{\mu\nu}$ and ϕ

$$\phi R_{\mu\nu} - \frac{\phi R + V(\phi) + \omega(\phi) \phi_{;\alpha} \phi^{;\alpha}}{2} g_{\mu\nu} + \omega(\phi) \phi_{;\mu} \phi_{;\nu} - \phi_{;\mu\nu} + g_{\mu\nu} \square \phi = \mathcal{X} T_{\mu\nu}$$

$$2\omega(\phi) \square \phi + \omega_{;\alpha}(\phi) \phi_{;\alpha} \phi^{;\alpha} - R V_{\phi}(\phi) = 0$$

$$\phi R + 2V(\phi) + \omega(\phi) \phi_{;\alpha} \phi^{;\alpha} - 3\square \phi = -\mathcal{X} T$$

- The solutions in the case of pointlike source are

$$\Phi(\mathbf{x}) = \frac{GM}{\phi^{(0)} |\mathbf{x}|} \left\{ 1 - \frac{e^{-m_{\phi} |\mathbf{x}|}}{2\omega(\phi^{(0)}) \phi^{(0)} - 3} \right\} \quad \Psi(\mathbf{x}) = -\frac{GM}{\phi^{(0)} |\mathbf{x}|} \left\{ 1 + \frac{e^{-m_{\phi} |\mathbf{x}|}}{2\omega(\phi^{(0)}) \phi^{(0)} - 3} \right\}$$

$$\phi(\mathbf{x}) = \phi^{(0)} - \frac{1}{2\omega(\phi^{(0)}) \phi^{(0)} - 3} \frac{r_g}{|\mathbf{x}|} e^{-m_{\phi} |\mathbf{x}|} \quad m_{\phi}^2 \doteq -\frac{\phi^{(0)} V_{\phi\phi}(\phi^{(0)})}{2\omega(\phi^{(0)}) \phi^{(0)} - 3}$$

The Scalar-Tensor Gravity in the Einstein frame and Conformal Transformations

- The action of the scalar-tensor theory of gravity in the Einstein frame

$$\mathcal{A}^{EF} = \int d^4x \sqrt{-\tilde{g}} \left[\Xi \tilde{R} + W(\tilde{\phi}) + \tilde{\omega}(\tilde{\phi}) \tilde{\phi}_{;\alpha} \tilde{\phi}^{;\alpha} + \mathcal{X} \tilde{\mathcal{L}}_m \right]$$

- The field equations for the fields $\tilde{g}_{\mu\nu}$ and $\tilde{\phi}$

$$\Xi \tilde{R}_{\mu\nu} - \frac{\Xi \tilde{R} + W(\tilde{\phi}) + \tilde{\omega}(\tilde{\phi}) \tilde{\phi}_{;\alpha} \tilde{\phi}^{;\alpha}}{2} \tilde{g}_{\mu\nu} + \tilde{\omega}(\tilde{\phi}) \tilde{\phi}_{;\mu} \tilde{\phi}_{;\nu} = \mathcal{X} \tilde{T}_{\mu\nu}$$

$$2 \tilde{\omega}(\tilde{\phi}) \tilde{\square} \tilde{\phi} + \tilde{\omega}_{;\bar{\alpha}}(\tilde{\phi}) \tilde{\phi}_{;\alpha} \tilde{\phi}^{;\alpha} - W_{;\bar{\alpha}}(\tilde{\phi}) - \mathcal{X} \frac{\delta \tilde{\mathcal{L}}_m}{\delta \tilde{\phi}} = 0$$

$$\Xi \tilde{R} + 2W(\tilde{\phi}) + \tilde{\omega}(\tilde{\phi}) \tilde{\phi}_{;\alpha} \tilde{\phi}^{;\alpha} = -\mathcal{X} \tilde{T}$$

- By imposing the conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$

$$\tilde{\omega}(\tilde{\phi}) d\tilde{\phi}^2 = \frac{\Xi}{2} [2\phi \omega(\phi) - 3] \left(\frac{d\phi}{\phi} \right)^2, \quad W(\tilde{\phi}) = \frac{\Xi^2}{\phi(\tilde{\phi})^2} V(\phi(\tilde{\phi})),$$

$$\tilde{\mathcal{L}}_m = \frac{\Xi^2}{\phi(\tilde{\phi})^2} \mathcal{L}_m \left(\frac{\Xi \tilde{g}_{\rho\sigma}}{\phi(\tilde{\phi})} \right), \quad \phi \Omega^{-2} = \Xi$$

The Newtonian limit of the Scalar Tensor Gravity in the Einstein frame

- If $\tilde{\omega}(\tilde{\phi}) = -1/2$, $\Xi = 1$ and $\omega(\phi) = -\omega_0/\phi$ the links between the two frames in the newtonian limit are

$$\tilde{\phi}(\phi) = \tilde{\phi}_0 + \sqrt{2\omega_0 + 3} \ln \phi, \quad W(\tilde{\phi}) = \exp\left(-\frac{2\tilde{\phi}}{\sqrt{2\omega_0 + 3}}\right) V\left(e^{\frac{\tilde{\phi}}{\sqrt{2\omega_0 + 3}}}\right)$$

$$\tilde{\mathcal{L}}_m = 2\rho \exp\left(-\frac{2\tilde{\phi}}{\sqrt{2\omega_0 + 3}}\right), \quad \phi \Omega^{-2} = 1$$

- The solutions generated again by the pointlike source

$$\tilde{\phi} = -\frac{GM}{|\mathbf{x}|} \quad \tilde{\Psi} = \tilde{\phi}$$

$$\tilde{\phi} = \sqrt{2\omega_0 + 3} \ln \phi^{(0)} + \frac{1}{\phi^{(0)} \sqrt{2\omega_0 + 3}} \frac{r_g}{|\mathbf{x}|} e^{-m_\phi |\mathbf{x}|}$$

- The difference between the gravitational potentials in the Jordan and Einstein frame is obtained

$$\tilde{\phi} - \phi^{(0)} \phi = \frac{\varphi}{2} = \frac{GM}{2\omega_0 + 3} \frac{e^{-m_\phi |\mathbf{x}|}}{|\mathbf{x}|}$$

Conclusions

- Any $f(R)$ -Gravity contributes to the more attractive force, but the Ricci tensor invariant contributes to the anti-Gravity: we have a lower theoretical rotation velocity (with respect to the pure $f(R)$ -Gravity), while the experimental evidence says opposite.

Conclusions

- Any $f(R)$ -Gravity contributes to the more attractive force, but the Ricci tensor invariant contributes to the anti-Gravity: we have a lower theoretical rotation velocity (with respect to the pure $f(R)$ -Gravity), while the experimental evidence says opposite.
- From the point of view of gravitational lensing we have a perfect agreement between $f(R)$ -Gravity and General Relativity in the case of thin lens. Only by adding $f(R_{\rho\sigma}R^{\rho\sigma})$ in the action we induce the modifications, but we do not find the hoped behavior.

Conclusions

- Any $f(R)$ -Gravity contributes to the more attractive force, but the Ricci tensor invariant contributes to the anti-Gravity: we have a lower theoretical rotation velocity (with respect to the pure $f(R)$ -Gravity), while the experimental evidence says opposite.
- From the point of view of gravitational lensing we have a perfect agreement between $f(R)$ -Gravity and General Relativity in the case of thin lens. Only by adding $f(R_{\rho\sigma}R^{\rho\sigma})$ in the action we induce the modifications, but we do not find the hoped behavior.
- In the galactic dynamics we studied the motion of massive particles and in this case we find the corrections induced also by only $f(R)$ -Gravity.

Conclusions

- Any $f(R)$ -Gravity contributes to the more attractive force, but the Ricci tensor invariant contributes to the anti-Gravity: we have a lower theoretical rotation velocity (with respect to the pure $f(R)$ -Gravity), while the experimental evidence says opposite.
- From the point of view of gravitational lensing we have a perfect agreement between $f(R)$ -Gravity and General Relativity in the case of thin lens. Only by adding $f(R_{\rho\sigma}R^{\rho\sigma})$ in the action we induce the modifications, but we do not find the hoped behavior.
- In the galactic dynamics we studied the motion of massive particles and in this case we find the corrections induced also by only $f(R)$ -Gravity.
- The galactic rotation curves need a Dark Matter component, but the choice of framework is crucial for the space model of Dark Matter.

Conclusions

- Any $f(R)$ -Gravity contributes to the more attractive force, but the Ricci tensor invariant contributes to the anti-Gravity: we have a lower theoretical rotation velocity (with respect to the pure $f(R)$ -Gravity), while the experimental evidence says opposite.
- From the point of view of gravitational lensing we have a perfect agreement between $f(R)$ -Gravity and General Relativity in the case of thin lens. Only by adding $f(R_{\rho\sigma}R^{\rho\sigma})$ in the action we induce the modifications, but we do not find the hoped behavior.
- In the galactic dynamics we studied the motion of massive particles and in this case we find the corrections induced also by only $f(R)$ -Gravity.
- The galactic rotation curves need a Dark Matter component, but the choice of framework is crucial for the space model of Dark Matter.
- Moreover if we consider $f(R, R_{\rho\sigma}R^{\rho\sigma})$ -Gravity for the gravitational lensing **we need a bigger amount of Dark Matter then in General Relativity.**

Conclusions

- Any $f(R)$ -Gravity contributes to the more attractive force, but the Ricci tensor invariant contributes to the anti-Gravity: we have a lower theoretical rotation velocity (with respect to the pure $f(R)$ -Gravity), while the experimental evidence says opposite.
- From the point of view of gravitational lensing we have a perfect agreement between $f(R)$ -Gravity and General Relativity in the case of thin lens. Only by adding $f(R_{\rho\sigma}R^{\rho\sigma})$ in the action we induce the modifications, but we do not find the hoped behavior.
- In the galactic dynamics we studied the motion of massive particles and in this case we find the corrections induced also by only $f(R)$ -Gravity.
- The galactic rotation curves need a Dark Matter component, but the choice of framework is crucial for the space model of Dark Matter.
- Moreover if we consider $f(R, R_{\rho\sigma}R^{\rho\sigma})$ -Gravity for the gravitational lensing **we need a bigger amount of Dark Matter then in General Relativity.**
- The behavior of anti-Gravity is similar to the Sanders potential.

Conclusions

- Any $f(R)$ -Gravity contributes to the more attractive force, but the Ricci tensor invariant contributes to the anti-Gravity: we have a lower theoretical rotation velocity (with respect to the pure $f(R)$ -Gravity), while the experimental evidence says opposite.
- From the point of view of gravitational lensing we have a perfect agreement between $f(R)$ -Gravity and General Relativity in the case of thin lens. Only by adding $f(R_{\rho\sigma}R^{\rho\sigma})$ in the action we induce the modifications, but we do not find the hoped behavior.
- In the galactic dynamics we studied the motion of massive particles and in this case we find the corrections induced also by only $f(R)$ -Gravity.
- The galactic rotation curves need a Dark Matter component, but the choice of framework is crucial for the space model of Dark Matter.
- Moreover if we consider $f(R, R_{\rho\sigma}R^{\rho\sigma})$ -Gravity for the gravitational lensing **we need a bigger amount of Dark Matter then in General Relativity.**
- The behavior of anti-Gravity is similar to the Sanders potential.
- The weak conformal transformations: the correspondence of the solutions in the Newtonian limit between the Jordan and Einstein frame

Conclusions

- Any $f(R)$ -Gravity contributes to the more attractive force, but the Ricci tensor invariant contributes to the anti-Gravity: we have a lower theoretical rotation velocity (with respect to the pure $f(R)$ -Gravity), while the experimental evidence says opposite.
- From the point of view of gravitational lensing we have a perfect agreement between $f(R)$ -Gravity and General Relativity in the case of thin lens. Only by adding $f(R_{\rho\sigma}R^{\rho\sigma})$ in the action we induce the modifications, but we do not find the hoped behavior.
- In the galactic dynamics we studied the motion of massive particles and in this case we find the corrections induced also by only $f(R)$ -Gravity.
- The galactic rotation curves need a Dark Matter component, but the choice of framework is crucial for the space model of Dark Matter.
- Moreover if we consider $f(R, R_{\rho\sigma}R^{\rho\sigma})$ -Gravity for the gravitational lensing **we need a bigger amount of Dark Matter then in General Relativity.**
- The behavior of anti-Gravity is similar to the Sanders potential.
- The weak conformal transformations: the correspondence of the solutions in the Newtonian limit between the Jordan and Einstein frame
- The effective Lagrangian in the Newtonian Limit: **Scalar Tensor Fourth Order Gravity (STFOG)**

Conclusions

- Any $f(R)$ -Gravity contributes to the more attractive force, but the Ricci tensor invariant contributes to the anti-Gravity: we have a lower theoretical rotation velocity (with respect to the pure $f(R)$ -Gravity), while the experimental evidence says opposite.
- From the point of view of gravitational lensing we have a perfect agreement between $f(R)$ -Gravity and General Relativity in the case of thin lens. Only by adding $f(R_{\rho\sigma}R^{\rho\sigma})$ in the action we induce the modifications, but we do not find the hoped behavior.
- In the galactic dynamics we studied the motion of massive particles and in this case we find the corrections induced also by only $f(R)$ -Gravity.
- The galactic rotation curves need a Dark Matter component, but the choice of framework is crucial for the space model of Dark Matter.
- Moreover if we consider $f(R, R_{\rho\sigma}R^{\rho\sigma})$ -Gravity for the gravitational lensing **we need a bigger amount of Dark Matter then in General Relativity.**
- The behavior of anti-Gravity is similar to the Sanders potential.
- The weak conformal transformations: the correspondence of the solutions in the Newtonian limit between the Jordan and Einstein frame
- The effective Lagrangian in the Newtonian Limit: **Scalar Tensor Fourth Order Gravity (STFOG)**
- Finally we conclude that **Dark matter effect as a pure single geometric phenomenon remains a hard challenge.**

Conclusions

- Any $f(R)$ -Gravity contributes to the more attractive force, but the Ricci tensor invariant contributes to the anti-Gravity: we have a lower theoretical rotation velocity (with respect to the pure $f(R)$ -Gravity), while the experimental evidence says opposite.
- From the point of view of gravitational lensing we have a perfect agreement between $f(R)$ -Gravity and General Relativity in the case of thin lens. Only by adding $f(R_{\rho\sigma}R^{\rho\sigma})$ in the action we induce the modifications, but we do not find the hoped behavior.
- In the galactic dynamics we studied the motion of massive particles and in this case we find the corrections induced also by only $f(R)$ -Gravity.
- The galactic rotation curves need a Dark Matter component, but the choice of framework is crucial for the space model of Dark Matter.
- Moreover if we consider $f(R, R_{\rho\sigma}R^{\rho\sigma})$ -Gravity for the gravitational lensing **we need a bigger amount of Dark Matter then in General Relativity.**
- The behavior of anti-Gravity is similar to the Sanders potential.
- The weak conformal transformations: the correspondence of the solutions in the Newtonian limit between the Jordan and Einstein frame
- The effective Lagrangian in the Newtonian Limit: **Scalar Tensor Fourth Order Gravity (STFOG)**
- Finally we conclude that **Dark matter effect as a pure single geometric phenomenon remains a hard challenge.**

Conclusions

- Any $f(R)$ -Gravity contributes to the more attractive force, but the Ricci tensor invariant contributes to the anti-Gravity: we have a lower theoretical rotation velocity (with respect to the pure $f(R)$ -Gravity), while the experimental evidence says opposite.
- From the point of view of gravitational lensing we have a perfect agreement between $f(R)$ -Gravity and General Relativity in the case of thin lens. Only by adding $f(R_{\rho\sigma}R^{\rho\sigma})$ in the action we induce the modifications, but we do not find the hoped behavior.
- In the galactic dynamics we studied the motion of massive particles and in this case we find the corrections induced also by only $f(R)$ -Gravity.
- The galactic rotation curves need a Dark Matter component, but the choice of framework is crucial for the space model of Dark Matter.
- Moreover if we consider $f(R, R_{\rho\sigma}R^{\rho\sigma})$ -Gravity for the gravitational lensing **we need a bigger amount of Dark Matter then in General Relativity.**
- The behavior of anti-Gravity is similar to the Sanders potential.
- The weak conformal transformations: the correspondence of the solutions in the Newtonian limit between the Jordan and Einstein frame
- The effective Lagrangian in the Newtonian Limit: **Scalar Tensor Fourth Order Gravity (STFOG)**
- Finally we conclude that **Dark matter effect as a pure single geometric phenomenon remains a hard challenge.**

Thanks for your attention