Probing Models of Extended Gravity using Gravity Probe B and LARES experiments

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Abstract

We consider models of Extended Gravity and in particular, generic models containing scalar-tensor and higher-order curvature terms, as well as a model derived from noncommutative spectral geometry. Studying, in the weak-field approximation (the Newtonian and Post-Newtonian limit of the theory), the geodesic and Lense-Thirring processions, we impose constraints on the free parameters of such models by using the recent experimental results of the Gravity Probe B and LARES satellites.

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References

G. Lambiase, M. Sakellariadou and An. Stabile *Constraints on NonCommutative Spectral Action from Gravity Probe B and Torsion Balance Experiments, JCAP12(2013)020*

S. Capozziello, G. Lambiase, M. Sakellariadou, A. Stabile, An. Stabile *Constraining Models of Extended Gravity using Gravity Probe B and LARES* experiments Submitted to Phys. Rev. D (2014)

Scalar Tensor Fourth Order Gravity (STFOG)

The action

$$\mathcal{A} = \int d^4x\sqrt{-g}\Big[f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} + \mathcal{X}\mathcal{L}_{matter}\Big]$$

- $R \rightarrow$ Ricci scalar
- $R_{\alpha\beta}R^{\alpha\beta} \doteq Y \rightarrow$Ricci tensor square
- $\phi \rightarrow$ scalar field
- $\mathcal{L}_{matter} \rightarrow$ minimally coupled ordinary matter
- $g \rightarrow$ determinant of metric tensor $g_{\mu\nu}$
- $\mathcal{X} = 8\pi G/c^4$ but we use $c = 1$
- $R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$, $R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} + \ldots \rightarrow$ convention
- $2\Gamma^{\mu}_{\alpha\beta} = g^{\mu\sigma}(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma})$
- $(+ - - -) \rightarrow$ the adopted signature

The field equations by applying $\delta(\cdot) \rightarrow \delta g_{\mu\nu} \frac{\delta(\cdot)}{\delta g_{\mu\nu}} + \delta\phi \frac{\delta(\cdot)}{\delta\phi}$

$$f_R R_{\mu\nu} - \frac{f + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}}{2} g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu}\Box f_R + 2f_Y R_\mu{}^\alpha R_{\alpha\nu} - 2[f_Y R^\alpha{}_{(\mu}]_{;\nu)\alpha}$$
$$+\Box[f_Y R_{\mu\nu}] + [f_Y R_{\alpha\beta}]^{;\alpha\beta} g_{\mu\nu} + \omega(\phi)\phi_{;\mu}\phi_{;\nu} = \mathcal{X}\, T_{\mu\nu}$$

$$2\omega(\phi)\Box\phi + \omega_\phi(\phi)\phi_{;\alpha}\phi^{;\alpha} - f_\phi = 0$$

$$f_R R + 2f_Y R_{\alpha\beta}R^{\alpha\beta} - 2f + \Box[3f_R + f_Y R] + 2[f_Y R^{\alpha\beta}]_{;\alpha\beta} - \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} = \mathcal{X}\, T$$

- $T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ $\rightarrow$ the tensor of matter
- $T = T^\rho{}_\rho$ $\rightarrow$ trace of matter tensor
- $f_R = \frac{df}{dR}$, $f_Y = \frac{df}{dY}$, $\omega_\phi = \frac{d\omega}{d\phi}$, $f_\phi = \frac{df}{d\phi}$, $\Box = {}_{;\sigma}{}^{;\sigma}$

The scalar field ϕ is coupled with the geometry R, $R_{\mu\nu}$ (as the classical scalar tensor gravity) and the field equations are differential equations of fourth order.

The tecnically possible frameworks of STFOG

The first two interesting frameworks are

Newtonian and Post-Newtonian limit	Weak field limit
$g_{\mu\nu} \sim \begin{pmatrix} 1 + g^{(2)}_{tt}(t,\mathbf{x}) + g^{(4)}_{tt}(t,\mathbf{x}) & g^{(3)}_{ti}(t,\mathbf{x}) \\ g^{(3)}_{ti}(t,\mathbf{x}) & -\delta_{ij} + g^{(2)}_{ij}(t,\mathbf{x}) \end{pmatrix}$	$g_{\mu\nu} \sim \eta_{\mu\nu} + h_{\mu\nu}(x)$
$\phi \sim \phi^{(0)} + \phi^{(2)}(t,\mathbf{x})$	$\phi \sim \phi^{(0)} + \phi^{(1)}(x)$
$;\mu \sim ,\mu + \tilde{;}\mu = -\nabla + \tilde{\partial}_\mu \qquad \Box \sim -\triangle + \tilde{\Box}$	$;\mu \sim ,\mu = \partial_\mu \qquad \Box \sim \Box_\eta$
Slow motion in the spherically symmetric systems	Gravitational waves

- $x = (t, \mathbf{x})$
- The approximation level is driven by an expansion of the powers of $1/c$
- The approximation level is driven by the linearization of field equations $\mathcal{O}(h_{\mu\nu})^2 \ll 1$
- In the Newtonian limit the space time is time independet

The Newtonian limit of STFOG

By setting $f_R(0,0,\phi^{(0)}) = 1$, $\omega(\phi^{(0)}) = 1/2$ the field equations are

$$(\triangle - m_Y{}^2)\triangle\Phi + \left[\frac{m_Y{}^2}{2} - \frac{m_R{}^2+2m_Y{}^2}{6m_R{}^2}\triangle\right]R + m_Y{}^2\, f_{R\phi}(0,0,\phi^{(0)})\,\triangle\varphi = -m_Y{}^2\mathcal{X}\, T_{tt}$$

$$(\triangle - m_R{}^2)R - 3m_R{}^2\, f_{R\phi}(0,0,\phi^{(0)})\,\triangle\varphi = m_R{}^2\,\mathcal{X}\, T$$

$$(\triangle - m_\phi{}^2)\varphi + f_{R\phi}(0,0,\phi^{(0)})\, R = 0$$

where

$$g_{\mu\nu} \sim \begin{pmatrix} 1+2\Phi(\mathbf{x}) & 0 \\ 0 & -\delta_{ij} \end{pmatrix}$$

$$\phi \sim \phi^{(0)} + \varphi$$

The derivatives are calculated on the Minkowskian background

$$m_R{}^2 \doteq -\frac{f_R(0,0,\phi^{(0)})}{3f_{RR}(0,0,\phi^{(0)})+2f_Y(0,0,\phi^{(0)})}$$

$$m_Y{}^2 \doteq \frac{f_R(0,0,\phi^{(0)})}{f_Y(0,0,\phi^{(0)})}$$

$$m_\phi{}^2 \doteq -f_{\phi\phi}(0,0,\phi^{(0)})$$

- $T_{tt} = T = \rho(\mathbf{x}) \rightarrow$ the energy mass density of the source
- $T = T^\rho{}_\rho \rightarrow$ trace of matter tensor
- **The theory is parameterized by the coefficients of Taylor expansion of Lagrangian.**

The Newtonian limit of STFOG: the pointlike gravitational potential

The scalar field φ and also the Ricci scalar R are auxiliary fields. The physical outcome is

$$\Phi(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}\left[1+g(\xi,\eta)\,e^{-m_R\tilde{k}_R|\mathbf{x}|}+[1/3-g(\xi,\eta)]\,e^{-m_R\tilde{k}_\phi|\mathbf{x}|}-\frac{4}{3}\,e^{-m_Y|\mathbf{x}|}\right]$$

where $g(\xi,\eta) = \frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}$, $\tilde{k}^2_{R,\phi} = \frac{1-\xi+\eta^2\pm\sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}$, $\eta = \frac{m_\phi}{m_R}$ and $\xi = 3f_{R\phi}(0,0,\phi^{(0)})^2$.

The effective Lagrangian of STFOG is given as

$$R+\frac{f_{RR}(0,0,\phi^{(0)})}{2}\,R^2+\frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2}\,\varphi^2+f_{R\phi}(0,0,\phi^{(0)})R\,\phi+f_Y(0,0,\phi^{(0)})R_{\alpha\beta}R^{\alpha\beta}+\frac{|\nabla\varphi|^2}{2}$$

...and if we add the terms proportional to $R_{\alpha\beta\gamma\beta}R^{\alpha\beta\gamma\delta}$ and/or to $\Box^k R$? **The outcome is the same**!!

- The Gauss-Bonnet invariant ...
- $\Box^k$ is the four divergence ...

The Newtonian limit of STFOG: Classes of the gravitational potentials (A, B)

Case	Gravitational potential	Free parameters
A	$-\frac{GM}{\lvert\mathbf{x}\rvert}\left[1+\frac{1}{3}\,e^{-m_R\lvert\mathbf{x}\rvert}\right]$	$m_R{}^2 = -\frac{f_R(0)}{3f_{RR}(0)}$
B	$-\frac{GM}{\lvert\mathbf{x}\rvert}\left[1+\frac{1}{3}\,e^{-m_R\lvert\mathbf{x}\rvert}-\frac{4}{3}\,e^{-m_Y\lvert\mathbf{x}\rvert}\right]$	$m_R{}^2 = -\frac{f_R(0,0)}{3f_{RR}(0,0)+2f_Y(0,0)}$ $m_Y{}^2 = \frac{f_R(0,0)}{f_Y(0,0)}$

- case A $\rightarrow f(R)$
- case B $\rightarrow f(R,\ R_{\alpha\beta}R^{\alpha\beta})$
- case C $\rightarrow f(R,\ \phi)+\omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$
- case D $\rightarrow f(R,\ R_{\alpha\beta}R^{\alpha\beta},\ \phi)+\omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$

The Newtonian limit of STFOG: Classes of the gravitational potentials (C)

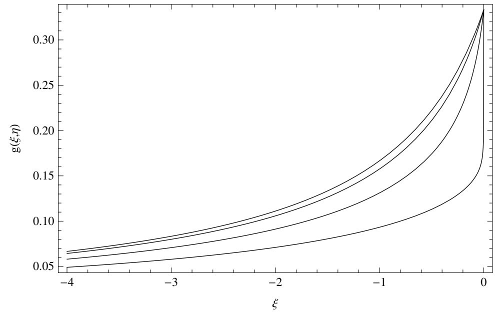
Gravitational potential	Free parameters
$-\frac{GM}{\lvert\mathbf{x}\rvert}\Big[1+g(\xi,\eta)\,e^{-m_R\tilde{k}_R\,\lvert\mathbf{x}\rvert}+$ $+[1/3-g(\xi,\eta)]\,e^{-m_R\tilde{k}_\phi\,\lvert\mathbf{x}\rvert}\Big]$	$m_R{}^2 = -\frac{f_R(0,\phi^{(0)})}{3f_{RR}(0,\phi^{(0)})}$ $m_\phi{}^2 = -\frac{f_{\phi\phi}(0,\phi^{(0)})}{2\omega(\phi^{(0)})}$ $\xi = \frac{3f_{R\phi}(0,\phi^{(0)})^2}{2\omega(\phi^{(0)})}$ $\eta = \frac{m_\phi}{m_R}$ $g(\xi,\,\eta) = \frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}$ $\tilde{k}^2_{R,\phi} = \frac{1-\xi+\eta^2\pm\sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}$

The Newtonian limit of STFOG: Classes of the gravitational potentials (D)

Gravitational potential	Free parameters
$-\frac{GM}{\vert\mathbf{x}\vert}\Big[1+g(\xi,\eta)\,e^{-m_R\tilde{k}_R\,\vert\mathbf{x}\vert}+$ $+[1/3-g(\xi,\eta)]\,e^{-m_R\tilde{k}_\phi\,\vert\mathbf{x}\vert}-\frac{4}{3}\,e^{-m_Y\vert\mathbf{x}\vert}\Big]$	$m_R{}^2 = -\frac{f_R(0,0,\phi^{(0)})}{3f_{RR}(0,0,\phi^{(0)})+2f_Y(0,0,\phi^{(0)})}$ $m_Y{}^2 = \frac{f_R(0,0,\phi^{(0)})}{f_Y(0,0,\phi^{(0)})}$ $m_\phi{}^2 = -\frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2\omega(\phi^{(0)})}$ $\xi = \frac{3f_{R\phi}(0,0,\phi^{(0)})^2}{2\omega(\phi^{(0)})}$ $\eta = \frac{m_\phi}{m_R}$ $g(\xi,\,\eta) =$ $= \frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}$ $\tilde{k}^2_{R,\phi} = \frac{1-\xi+\eta^2\pm\sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}$

The Newtonian limit of STFOG: The crucial choice of the coefficient and of the theory!

The potential does not depend only on the masses propagating (cases A, B) but also on the function modeling the Yukawa corrections (case C, D).



Plot of coefficient $g(\xi,\eta)$ (case C) with respect to quantity ξ for $0 \leq \eta \leq 0.99$ with step 0.33.
$g(\xi,\eta) = \frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}$, $\tilde{k}^2_{R,\phi} = \frac{1-\xi+\eta^2\pm\sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}$, $\eta = \frac{m_\phi}{m_R}$ and $\xi = \frac{3f_{R\phi}(0,0,\phi^{(0)})^2}{2\omega(\phi^{(0)})}$.

The Newtonian limit of STFOG: the Gauss theorem is not verified!

- **In STFOG the Gauss theorem is not verified!**
- The case of source ball-like is different from the pointlike one: the potential depends on the dimension of the source.
- Generally for any term $\propto \frac{e^{-mr}}{r}$ we have a geometric factor multipliyng the Yukawa term.
- In the case of a ball with radius $\mathcal{R}$ we find $F(m\mathcal{R}) = 3\frac{m\mathcal{R}\cosh m\mathcal{R} - \sinh m\mathcal{R}}{m^3\mathcal{R}^3}$

$$\Phi_{ball}(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}\left[1 + g(\xi,\eta)\,F(m_R\tilde{k}_R\mathcal{R})\,e^{-m_R\tilde{k}_R|\mathbf{x}|} + [1/3 - g(\xi,\eta)]\,F(m_R\tilde{k}_\phi\mathcal{R})\,e^{-m_R\tilde{k}_\phi|\mathbf{x}|} - \frac{4}{3}\,F(m_Y\mathcal{R})\,e^{-m_Y|\mathbf{x}|}\right]$$

The Post-Newtonian limit of STFOG

- Post-Newtonian field equations:

$$\left\{(\triangle - m_Y{}^2)\triangle\Psi - \left[\frac{m_Y{}^2}{2} - \frac{m_R{}^2+2m_Y{}^2}{6m_R{}^2}\triangle\right]R - m_Y{}^2\, f_{R\phi}(0,0,\phi^{(0)})\,\triangle\varphi\right\}\delta_{ij}$$

$$+\left\{(\triangle - m_Y{}^2)(\Psi - \Phi) + \frac{m_R{}^2 - m_Y{}^2}{3m_R{}^2}\,R + m_Y{}^2\, f_{R\phi}(0,0,\phi^{(0)})\,\varphi\right\}_{,ij} = 0\,,$$

$$\left\{(\triangle - m_Y{}^2)\triangle A_i + m_Y{}^2\mathcal{X}\,\rho\, v_i\right\} + \left\{(\triangle - m_Y{}^2)\Psi + \frac{m_R{}^2 - m_Y{}^2}{3m_R{}^2}R + m_Y{}^2\, f_{R\phi}(0,0,\phi^{(0)})\,\varphi\right\}_{,ti}$$

- The solutions for a ball with radius $\mathcal{R}$ are:

$$\Psi_{ball}(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}\left[1 - g(\xi,\eta)\,F(m_R\tilde{k}_R\mathcal{R})\,e^{-m_R\tilde{k}_R|\mathbf{x}|} - \frac{2\,F(m_Y\mathcal{R})}{3}\,e^{-m_Y|\mathbf{x}|} + \right.$$

$$\left. -[\frac{1}{3} - g(\xi,\eta)]\,F(m_R\tilde{k}_\phi\mathcal{R})\,e^{-m_R\tilde{k}_\phi|\mathbf{x}|}\right]\,.$$

$$\mathbf{A}(\mathbf{x}) = \frac{G}{|\mathbf{x}|^2}\left[1 - (1 + m_Y|\mathbf{x}|)\,e^{-m_Y|\mathbf{x}|}\right]\hat{\mathbf{x}}\times\mathbf{J}\,,$$

summarizing

- The metric is:

$$
\begin{aligned}
g_{tt} &= 1 + 2\Phi_{ball}(\mathbf{x}) \\
g_{ti} &= 2A_i(\mathbf{x}) \\
g_{ij} &= -\delta_{ij} + 2\Psi_{ball}(\mathbf{x})\delta_{ij}
\end{aligned}
$$

- The non-vanishing Christoffel symbols are:

$$
\begin{aligned}
\Gamma^t_{ti} &= \Gamma^i_{tt} = \partial_i\Phi_{\rm ball} \; , \\
\Gamma^i_{tj} &= \frac{\partial_i A_j - \partial_j A_i}{2} \; , \\
\Gamma^i_{jk} &= \delta_{jk}\partial_i\Psi_{\rm ball} - \delta_{ij}\partial_k\Psi_{ball} - \delta_{ik}\partial_j\Psi_{\rm ball} \; .
\end{aligned}
$$

Rotating sources and orbital parameters

- Let us consider the geodesic equations:

$$\frac{d^2\,x^\mu}{ds^2}+\Gamma^\mu_{\alpha\beta}\,\frac{dx^\alpha}{ds}\,\frac{dx^\beta}{ds}\;=\;0$$

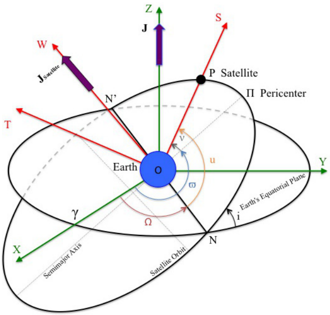
- We get the accelerations:

$$\begin{aligned}
\ddot{x}+\frac{GM}{r^3}x\;&=\;-\frac{GM\Lambda(r)}{r^3}x+\frac{2GJ}{r^5}\Big\{\zeta(r)\Big[\Big(x^2+y^2-2z^2\Big)\dot{y}+3yz\dot{z}\Big]+2\Sigma(r)L_x z\Big\}\\
\ddot{y}+\frac{GM}{r^3}y\;&=\;-\frac{GM\Lambda(r)}{r^3}y-\frac{2GJ}{r^5}\Big\{\zeta(r)\Big[\Big(x^2+y^2-2z^2\Big)\dot{x}+3xz\dot{z}\Big]-2\Sigma(r)L_y z\Big\}\\
\ddot{z}+\frac{GM}{r^3}z\;&=\;-\frac{GM\Lambda(r)}{r^3}z+\frac{6GJ}{r^5}\Big\{\zeta(r)+\frac{2}{3}\Sigma(r)\Big\}L_z z
\end{aligned}$$

- Where L_x,L_y and L_z are the components of the angular momentum and...

$$\begin{aligned}
\Lambda(r)\;&\doteq\;g(\xi,\eta)\,F(m_R\tilde{k}_R\mathcal{R})\,(1+m_R\tilde{k}_R r)\,e^{-m_R\tilde{k}_R r}-\frac{4\,F(m_Y\mathcal{R})}{3}\,(1+m_Y r)\,e^{-m_Y r}+\\
&\qquad+[1/3-g(\xi,\eta)]\,F(m_R\tilde{k}_\phi\mathcal{R})\,(1+m_R\tilde{k}_\phi r)\,e^{-m_R\tilde{k}_\phi r}\\
\zeta(r)\;&\doteq\;1-\Big[1+m_Y r+(m_Y r)^2\Big]e^{-m_Y r}\\
\Sigma(r)\;&\doteq\;(m_Y r)^2 e^{-m_Y r}
\end{aligned}$$

Orbital parameters



Is useful to introduce the conventional astronomical notation: a semimajor axis; e eccentricity; $p = a(1 - e^2)$ semilatus rectum; i inclination; Ω longitude of the ascending node N; $\tilde{\omega}$ longitude of the pericenter Π; L_0, mean longitude of the epoch, i.e. longitude of the satellite at time $t = 0$ (also a broken angle, measured from the X axis); ν true anomaly; $u = \nu + \tilde{\omega} - \Omega$ argument of the latitude; $n = (GM/a^3)^{1/2}$ mean daily motion; $C = r^2\dot{\nu} = na^2(1 - e^2)^{1/2}$ twice the areal velocity.

Orbital parameters

- The transformation rules between the coordinates frames (X, Y, Z) and (S, T, W) are

$$\begin{aligned} x &= r(\cos u \cos\Omega - \sin u \sin\Omega \cos i) \\ y &= r(\cos u \sin\Omega + \sin u \cos\Omega \cos i) \\ z &= r \sin u \sin i \\ r &= \tfrac{p}{1+e\cos\nu} \end{aligned}$$

- The components of the perturbing acceleration in the $(S,\ T,\ W)$ system read

$$\begin{aligned} A_s &= -\frac{G\,M\,\Lambda(r)}{r^2} + \frac{2\,G\,J\,C\cos i}{r^4}\zeta(r) \\ A_t &= -\frac{2\,G\,J\,C\,e\cos\ i\sin\nu}{p\,r^3}\zeta(r) \\ A_w &= \frac{2\,G\,J\,C\sin i}{r^4}\left[\left(\frac{r\,e\sin\nu\cos u}{p} + 2\sin u\right)\zeta(r) + 2\sin u\,\Sigma(r)\right]\ . \end{aligned}$$

- The A_s component has two contributions: the former one results from the modified Newtonian potential $\Phi_{\rm ball}(\mathbf{x})$, while the latter one results from the gravito-magnetic field A_i and it is a higher order term than the first one. Note that the components A_t and A_w depend only on the gravito-magnetic field.

Gauss equations

- The Gauss equations for the variations of the six orbital parameters, resulting from the perturbing acceleration with components A_x, A_y, A_z, read:

$$\frac{da}{dt} = \dot{a}_{\rm STFOG} = \frac{2e\,G\,M\,\Lambda(r)\,\sin\nu}{n\sqrt{1-e^2}\,C}\,\dot{\nu}$$

$$\frac{de}{dt} = \dot{e}_{\rm GR} + \dot{e}_{\rm STFOG} = \frac{\sqrt{1-e^2}\,G\,M\,\Lambda(r)\,\sin\nu}{n\,a\,C}\,\dot{\nu} + \dot{e}_{\rm GR}\left[1 - e^{-m_Y r}\left(1 + m_Y r + (m_Y r)^2\right)\right]$$

$$\frac{d\Omega}{dt} = \dot{\Omega}_{\rm GR} + \dot{\Omega}_{\rm STFOG} = \dot{\Omega}_{\rm GR}\left\{1 - e^{-m_Y r}\left[1 + m_Y r + \left(1 + f(\nu, u, e)\right)(m_Y r)^2\right]\right\}$$

$$\frac{di}{dt} = \dot{i}_{\rm GR} + \dot{i}_{\rm STFOG} = \dot{i}_{\rm GR}\left\{1 - e^{-m_Y r}\left[1 + m_Y r + \left(1 + f(\nu, u, e)\right)(m_Y r)^2\right]\right\}$$

$$\frac{d\tilde{\omega}}{dt} = \dot{\tilde{\omega}}_{\rm GR} + \dot{\tilde{\omega}}_{\rm STFOG} = -\frac{\sqrt{1-e^2}\,G\,M\,\Lambda(r)\,\cos\nu}{n\,a\,e\,C}\,\dot{\nu} + \dot{\tilde{\omega}}_{\rm GR}\left[1 - e^{-m_Y r}\left(1 + m_Y r + (m_Y r)^2\right)\right] + -2\sin^2\frac{i}{2}\,\dot{\Omega}_{\rm GR}\,f(\nu, u, e)\Sigma(r)$$

$$\frac{dM^0}{dt} = \dot{M}^0{}_{\rm GR} + \dot{M}^0{}_{\rm STFOG} = -\frac{G\,M\,\Lambda(r)}{n\,a\,C}\left[\frac{2r}{a} + \frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\cos\nu\right]\dot{\nu} + +\dot{M}^0{}_{\rm GR}\left[1 - e^{-m_Y r}\left(1 + m_Y r + (m_Y r)^2\right)\right] - 2\sin^2\frac{i}{2}\,\dot{\Omega}_{\rm GR}\,f(\nu, u, e)\Sigma(r)$$

- where:

$$f(\nu, u, e) = \frac{1 + e\cos\nu}{1 + e\left(\frac{\sin\nu\cot u}{2} + \cos\nu\right)}$$

Gauss equations: secular terms

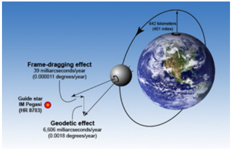
- Considering an almost circular orbit ($e \ll 1$), we integrate the Gauss equations with respect to the only anomaly ν, from 0 to $\nu(t) = nt$, since all other parameters have a slower evolution than ν, hence they can be considered as constraints with respect to ν. At first order we get

$$
\begin{aligned}
\Delta a(t) &= 0 \\
\Delta e(t) &= 0 \\
\Delta i(t) &= \frac{G\,J\,e^2\,\sin i}{na^3}\, e^{-m_Y p}(m_Y p)^2\left[1+\frac{(m_Y p)^2}{2}\left(m_Y p - 4\right)\right]\sin\left(\tilde{\omega}(t)-\Omega(t)\right)\nu(t)+\mathcal{O}(e^4) \\
\Delta\Omega(t) &= \frac{2\,G\,J}{na^3}\left[1-e^{-m_Y p}\left(1+m_Y p+2(m_Y p)^2\right)\right]\nu(t)+\mathcal{O}(e^2) \\
\Delta\tilde{\omega}(t) &= \left\{\frac{\tilde{\Lambda}(p)}{2}-\frac{2\,G\,J}{na^3}\left[3\cos i - 1 + e^{-m_Y p}(1+m_Y p+\tfrac{3}{2}(m_Y p)^2 + \right.\right. \\
&\qquad \left.\left. -(3+3m_Y p+3(m_Y p)^2+\tfrac{1}{12}(m_Y p)^3)\cos i)\right]\right\}\nu(t)+\mathcal{O}(e^2) \\
\Delta\mathcal{M}^0(t) &= \left\{2\Lambda(p)-\frac{2\,G\,J}{na^3}\left[3\cos i - 1 - e^{-m_Y p}\left(1+m_Y p+2(m_Y p)^2\right)\left(\cos i - 1\right)\right]\right\}\nu(t)+\mathcal{O}(e^2)
\end{aligned}
$$

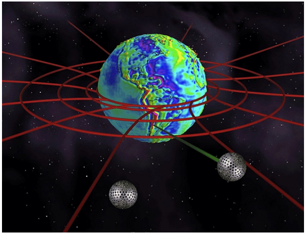
where

$$
\begin{aligned}
\tilde{\Lambda}(p) \doteq\; & g(\xi,\eta)\,F(m_R\tilde{k}_R\mathcal{R})\,(m_R\tilde{k}_R p)^2\,e^{-m_R\tilde{k}_R p}+[1/3-g(\xi,\eta)]\,F(m_R\tilde{k}_\phi\mathcal{R})\,(m_R\tilde{k}_\phi p)^2\,e^{-m_R\tilde{k}_\phi p} \\
& -\frac{4\,F(m_Y\mathcal{R})}{3}\,(m_Y p)^2\,e^{-m_Y p}\;.
\end{aligned}
$$

Gravity Prob B and LARES



	Measured	Predicted
Geodetic precession (mas)	6602 ± 18	6606
Frame-dragging (mas)	37.2 ± 7.2	39.2



- Gravity Prob B: was a relativity gyroscope experiment funded by NASA which launched on 20 April 2004 and and completed on 8 December 2010. The mission plans were to test two unverified predictions of general relativity: the geodetic effect and frame-dragging. This was to be accomplished by measuring, very precisely, tiny changes in the direction of spin of four gyroscopes contained in an Earth satellite orbiting at 650 *km* altitude, crossing directly over the poles.
- LARES: (acronym for Laser Relativity Satellite) is an Italian Space Agency scientific satellite launched on 13 February 2012.The satellite, completely passive, is made of tungsten alloy and houses 92 cube corner retro reflectors that are used to track the satellite via laser from stations on Earth. LARES's body has a diameter of about 36.4 *cm* and weighs about 400 *Kg*. LARES was inserted in an orbit with 1450 *Km* of perigee, an inclination of 69.5 degrees and reduced eccentricity $\sim 10^{-3}$. The satellite is tracked by the International Laser Ranging Service stations. The main scientific target of the LARES mission is the measurement of the frame-dragging, also known as Lense-Thirring effect, with an accuracy of about 1%.

Experimental constraints

- Geodesic and Lense-Thirring precessions are:

$$\mathbf{\Omega}_{\text{G}} = \frac{\nabla(\Phi+2\Psi)}{2}\times\mathbf{v} = \mathbf{\Omega}_{\text{G}}^{(\text{GR})}+\mathbf{\Omega}_{\text{G}}^{(\text{STFOG})}\,,\quad \mathbf{\Omega}_{\text{LT}} = \frac{\nabla\times\mathbf{A}}{2} = \mathbf{\Omega}_{\text{LT}}^{(\text{GR})}+\mathbf{\Omega}_{\text{LT}}^{(\text{STFOG})}$$

- Imposing the constraint $|\Omega| \lesssim \Omega^{(GR)} + \delta\Omega \Rightarrow |\Omega^{(STFOG)}| \lesssim \delta\Omega$, we get:
- Gravity Probe B

$$g(\xi,\eta)(m_R\tilde{k}_R\,r^*+1)\,F(m_R\tilde{k}_R R_\oplus)\,e^{-m_R\tilde{k}_R\,r^*}+\frac{8}{3}(m_Y\,r^*+1)\,F(m_Y R_\oplus)\,e^{-m_Y\,r^*}+$$
$$+[1/3-g(\xi,\eta)](m_R\tilde{k}_\phi\,r^*+1)\,F(m_R\tilde{k}_\phi R_\oplus)\,e^{-m_R\tilde{k}_\phi\,r^*} \lesssim \frac{3\,\delta|\mathbf{\Omega}_{\text{G}}|}{|\mathbf{\Omega}_{\text{G}}^{(\text{GR})}|} \simeq 0.008\,,$$

$$(1+m_Y r^*+{m_Y}^2 r^{*2})\,e^{-m_Y r^*} \lesssim \frac{\delta|\mathbf{\Omega}_{\text{LT}}|}{|\mathbf{\Omega}_{\text{LT}}^{(\text{GR})}|} \simeq 0.19 \quad\Rightarrow\quad m_Y \geq 7.3\times10^{-7}m^{-1}$$

- LARES

$$(1+m_Y r^*+{m_Y}^2 r^{*2})\,e^{-m_Y r^*} \lesssim \frac{\delta|\mathbf{\Omega}_{\text{LT}}|}{|\mathbf{\Omega}_{\text{LT}}^{(\text{GR})}|} \simeq 0.01 \quad\Rightarrow\quad m_Y \geq 1.2\times10^{-6}m^{-1}$$

$r^* = R_\oplus + h$, $R_\oplus$ is the radius of the Earth and h is the altitude of the satellite ($h = 650\ km$ for Gravity Probe B, while $h = 1450Km$ for LARES).

Noncommutative Spectral Geometry model

- Gravitational part of the asymptotic expression for the bosonic sector of the NCSG action:

$$\mathcal{S}_{\rm grav}^{\rm L} = \int {\rm d}^4 x \sqrt{-g}\left[\frac{R}{2\kappa_0^2} + \alpha_0 C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + \tau_0 R^\star R^\star - \xi_0 R|\mathbf{H}|^2\right]$$

- $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = 2R_{\alpha\beta}R^{\alpha\beta} - \frac{2}{3}R^2$.
- $\mathbf{H} = (\sqrt{af_0}/\pi)\phi$; $\alpha_0 = -3f_0/(10\pi^2)$; $\xi_0 = \frac{1}{12}$.
- $m_R \rightarrow \infty$; $m_Y = \sqrt{\frac{5\pi^2\left(k_0^2\mathbf{H}^{(0)}-6\right)}{36 f_0 k_0^2}}$; $m_\phi = 0$; $\xi = \frac{af_0(\mathbf{H}^{(0)})^2}{12\pi^2}$, $\eta = 0$;
  $g(\xi,\eta) = \frac{af_0(\mathbf{H}^{(0)})^2+12\pi^2}{6|af_0(\mathbf{H}^{(0)})^2-12\pi^2|} + \frac{1}{6}$; $\tilde{k}_{R,\phi}^2 = 1 - \frac{af_0(\mathbf{H}^{(0)})^2}{12\pi^2}$, 0.
- Geodesic effect: $\frac{8}{3}(m_Y\, r^* + 1)\, F(m_Y R_\oplus)\, e^{-m_Y\, r^*} \lesssim 0.008 \Rightarrow$, $m_Y \geq 7.1\times 10^{-5} m^{-1}$
- But... a more stringent constraint has been obtained using torsion balance experiments, $m_Y \geq 10^4 m^{-1}$

Conclusions

- In the context of Extended Gravity (STFOG), we have studied the linearized field equations in the limit of weak gravitational fields and small velocities generated by rotating gravitational sources, aiming at constraining the free parameters using recent recent experimental results.
- We have shown that the induced Extended Gravity effects depend on the effective masses m_R, m_Y and m_ϕ, while the nonvalidity of the Gauss theorem implies that these effects also depend on the geometric form and size of the rotating source.
- Requiring that the corrections are within the experimental errors, we then imposed constraints on the free parameters of the considered Extended Gravity model.
- The field Φ is time-independent. This aspect guarantees that A_i does not depend on the masses m_R and m_ϕ and, in the case of $f(R,\,\phi)$ gravity, the solution is the same as in GR.
- Merging the experimental results of Gravity Probe B and LARES, our results can be summarized as follows:

$$g(\xi,\eta)(m_R\tilde{k}_R\,r^*+1)\,F(m_R\tilde{k}_R R_\oplus)\,e^{-m_R\tilde{k}_R\,r^*}+\frac{8}{3}(m_Y\,r^*+1)\,F(m_Y R_\oplus)\,e^{-m_Y\,r^*}+$$
$$+[1/3-g(\xi,\eta)](m_R\tilde{k}_\phi\,r^*+1)\,F(m_R\tilde{k}_\phi R_\oplus)\,e^{-m_R\tilde{k}_\phi\,r^*}\lesssim 0.008$$

and

$$m_Y\geq 1.2\times 10^{-6}m^{-1}$$

Thanks for your attention