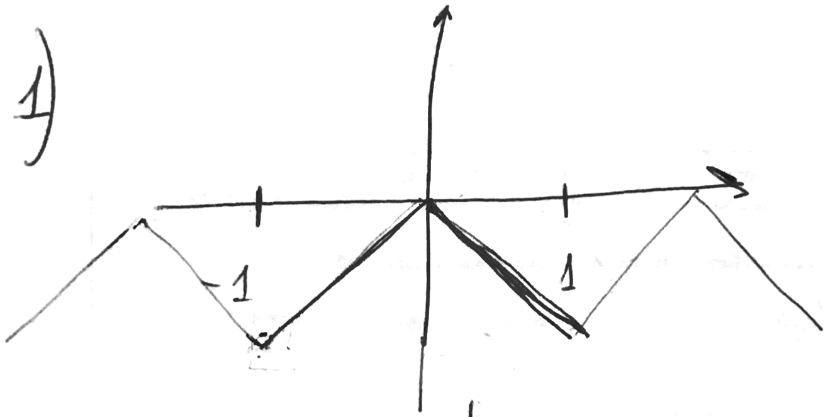
SVOLGIMENTO DELLA PROVA 4/7/2017 (A)

1)



$$f(x) = \begin{cases} x & -1 \le x < 0 \\ -x & 0 \le x < 1 \end{cases}$$

$T = 2$ $\qquad \omega = \frac{2\pi}{T} = \pi$

$f(x)$ è pari; $\quad b_n = 0$;

$$\frac{a_0}{2} = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\,dx = \frac{2}{T}\int_0^{T/2} f(x)\,dx = \frac{2}{2}\int_0^1 -x\,dx = -\frac{1}{2};$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} dx\, f(x)\cos n\omega x\,dx = -\frac{4}{T}\int_0^{T/2} dx\, x \cos n\omega x =$$

$$= -2\int_0^1 dx\, \underset{f}{x}\, \underset{g'}{\cos n\omega x} = -2\left[+\frac{x \sin n\omega x}{n\omega}\Big|_0^1 - \frac{1}{n\omega}\int_0^1 dx \sin n\omega x\right] =$$

$$= -2\left[\cancelto{0}{\frac{\sin n\omega}{n\omega}} + \frac{1}{n^2\omega^2}\cos n\omega x\Big|_0^1\right] = -2\,\frac{\cos n\omega - 1}{n^2\omega^2} = -2\,\frac{\cos n\pi - 1}{n^2\pi^2} =$$

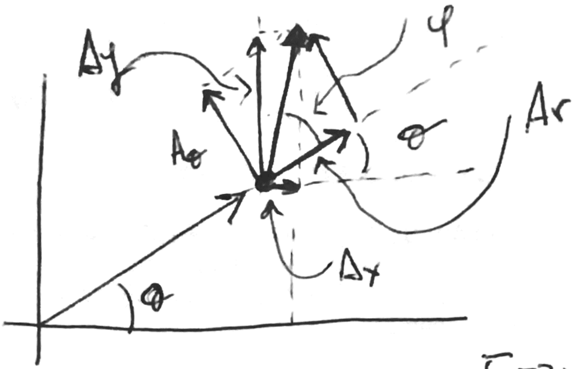
$$= -2\,\frac{(-1)^n - 1}{n^2\pi^2};$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega x$$

$$= -\frac{1}{2} - \sum_{n=1}^{\infty} 2\left[\frac{(-1)^n - 1}{n^2\pi^2}\right]\cos n\pi x;$$

$$= -\frac{1}{2} + \frac{4}{\pi^2}\cos \pi x + \frac{4}{9\pi^2}\cos 3\pi x + \ldots$$

B



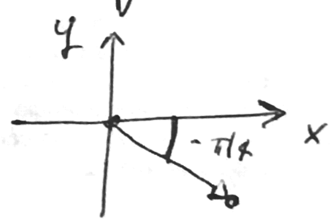
$$A_x = |\vec{A}| \cos(\theta + \varphi) = \left[|\vec{A}| \cos\varphi\right] \cos\theta - \left[|\vec{A}| \sin\varphi\right] \sin\theta$$

$$A_y = |\vec{A}| \sin(\theta + \varphi) = \left[|\vec{A}| \cos\varphi\right] \sin\theta + \left[|\vec{A}| \sin\varphi\right] \cos\theta$$

$$\begin{cases} A_x = A_r \cos\theta - A_\theta \sin\theta \\ A_y = A_r \sin\theta + A_\theta \cos\theta \end{cases}$$

$$\begin{cases} A_r = A_x \cos\theta + A_y \sin\theta \\ A_\theta = - A_x \sin\theta + A_y \cos\theta \end{cases}$$

$$\theta = \operatorname{arctg} \frac{r_y}{r_x} = \operatorname{arctg}(-1) = -\pi/4$$



$$A_x = 3(3\sqrt{2})^2 + 6(-3\sqrt{2}) = 54 - 18\sqrt{2}$$

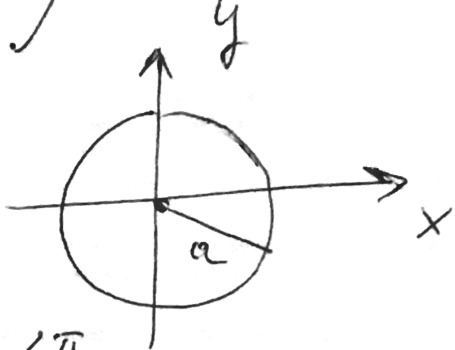
$$A_y = -14(-3\sqrt{2})(4) = 168\sqrt{2}$$

$$\begin{cases} A_r = (54 - 18\sqrt{2}) \frac{\sqrt{2}}{2} + 168\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) = 3(9\sqrt{2} - 62) \\ A_\theta = -(54 - 18\sqrt{2})\left(-\frac{\sqrt{2}}{2}\right) + 168\sqrt{2} \frac{\sqrt{2}}{2} = 3(50 + 9\sqrt{2}) \end{cases}$$

$$A_z = A_z = 20 \cdot 3\sqrt{2} = 60\sqrt{2}$$

$$\vec{A} = \left(3(9\sqrt{2} - 62),\ 3(9\sqrt{2} + 50),\ 60\sqrt{2}\right)$$

3) $\vec{F} = \left(-\dfrac{y}{(x^2+y^2)^2}, \dfrac{x}{(x^2+y^2)^2}, 1 \right)$ C



$\gamma : (a\cos t, a\sin t)$

$\gamma' = (-a\sin t, a\cos t)$

$$\oint \vec{F}\cdot d\vec{s} = \int_0^{2\pi} (F_x\,dx + F_y\,dy) = \int_0^{2\pi} \left[-\frac{a\sin t}{a^4}(-a\sin t)\,dt + \frac{a\cos t}{a^4}(+a\cos t)\,dt \right] =$$

$$= +\frac{1}{a^2}\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = +\frac{1}{a^2}\int_0^{2\pi} dt = +\frac{2\pi}{a^2}$$

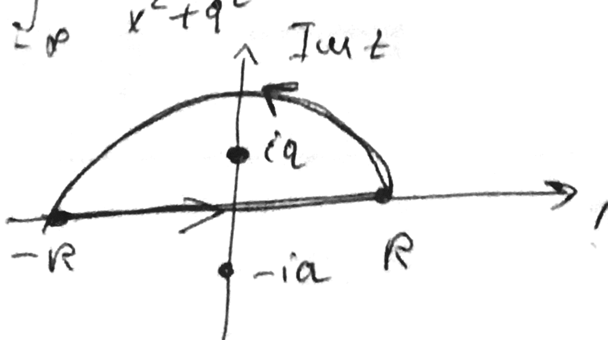
il risultato "dipende" dal raggio della circonferenza quindi la circuitazione dipende dalla particolare scelta della curva.

4) $f(\rho,\theta) = \rho^{1/2} e^{i\theta/2}$

$$\boxed{\frac{\partial f}{\partial \rho}} = \frac{1}{i\rho}\boxed{\frac{\partial f}{\partial \theta}} \qquad \boxed{\frac{1}{2\rho^{1/2}}} = \frac{1}{i\rho}\boxed{\rho^{1/2} e^{i\theta/2}\cdot\frac{i}{2}}$$

è olomorfa.

5) $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos mx}{x^2+a^2}\,dx \longrightarrow \oint \frac{e^{imz}}{z^2+a^2}\,dz = 2\pi i\,\mathrm{Res}(f, ia)$



$$\mathrm{Res}(f, ia) = \lim_{z\to ia} \frac{\cancel{(z-ia)}\,e^{imz}}{(z+ia)\cancel{(z-ia)}}$$

$$= \frac{e^{-ma}}{2ia}$$

I

D

$$\oint \frac{e^{imz}}{z^2+a^2}\,dz = \cancel{2}\pi \cancel{i}\,\frac{e^{-ma}}{\cancel{2i}a} = \frac{\pi}{a}e^{-ma}$$

$$\oint \frac{e^{imz}}{z^2+a^2}\,dz = \int_{-R}^{R} \frac{e^{imx}}{x^2+a^2}\,dx + \int_{\Gamma_R} \frac{e^{imz}}{z^2+a^2}\,dz$$

nel limite $z \to \infty$ $\int_{\Gamma_R} \to 0$ (TEOREMA GRANDE CERCHIO O LEMMA DI JORDAN)

$$\int_{-R}^{R} \frac{e^{imx}}{x^2+a^2}\,dx = \int_{-R}^{0} + \int_{0}^{R} = \underbrace{\int_{R}^{0} \frac{e^{-imx}(-dx)}{x^2+a^2}}_{x \to -x} + \int_{0}^{R} \frac{e^{imx}\,dx}{x^2+a^2} =$$

$$= \int_{0}^{R} \frac{e^{-imx}\,dx}{x^2+a^2} + \int_{0}^{R} \frac{e^{imx}}{x^2+a^2}\,dx = \int_{0}^{R} \frac{e^{imx}+e^{-imx}}{x^2+a^2}\,dx =$$

$$= 2\int_{0}^{R} \frac{\cos mx}{x^2+a^2}\,dx$$

per $R \to \infty$ otteniamo

$$2\int_{0}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \pi/a\, e^{-ma}$$

$$\int_{-\infty}^{+\infty} \frac{\cos mx}{x^2+a^2}\,dx = \pi/a\, e^{-ma}.$$

6) $y''' - y'' + y' = 1$

$s^3\hat{y} - s^2\hat{y} + s\hat{y} = \frac{1}{s}$ $\qquad \hat{y}(s^3 - s^2 + s) = \frac{1}{s}$

(E)

$$\hat{y} = \frac{1}{s^2(s^2-s+1)} \; ; \qquad s^2-s+1=0$$

$$s_{1,2} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2} \overset{!}{=} \alpha \pm i\beta$$

$$\alpha = 1/2 \; ; \; \beta = \sqrt{3}/2$$

$$\hat{y} = \frac{1}{s^2} \frac{1}{(s-s_1)(s-s_2)}$$

$$\frac{1}{(s-s_1)(s-s_2)} = \frac{A}{s-s_1} + \frac{B}{s-s_2} = \frac{(A+B)s - s_2 A - s_1 B}{(s-s_1)(s-s_2)}$$

$$\begin{cases} A+B=0 \\ -s_2 A - s_1 B = 1 \end{cases} \qquad \begin{cases} A = -B \\ (s_2 - s_1) B = 1. \end{cases} \qquad \begin{cases} A = \frac{1}{s_1-s_2} \\ B = \frac{1}{s_2-s_1} \end{cases}$$

$$\hat{y} = \frac{1}{s^2}\left[\frac{1}{s_1-s_2}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right)\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s-s_1}\right] = e^{s_1 t} = e^{\alpha t} e^{i\beta t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s-s_2}\right] = e^{s_2 t} = e^{\alpha t} e^{-i\beta t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s_1-s_2}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right)\right] = \frac{1}{\alpha+i\beta-(\alpha-i\beta)} e^{\alpha t}\left(e^{i\beta t} - e^{-i\beta t}\right)$$

$$= \frac{e^{\alpha t}}{\cancel{2i}\beta} \cancel{2i} \sin\beta t = \frac{e^{\alpha t}\sin\beta t}{\beta}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$$

$$y(t) = \mathcal{L}^{-1}[\hat{y}] = \frac{1}{\beta}\int_0^t d\tau \, \tau \, e^{\alpha(t-\tau)} \sin\beta(t-\tau) =$$

$$= \frac{e^{\alpha t}}{\beta} \int_0^t d\tau \, \tau \, e^{-\alpha \tau} \sin \beta (t-\tau) =$$

$$= 1 + t - e^{t/2}\left(\cos\left(\frac{\sqrt{3}}{2} t\right) + \frac{\sqrt{3}}{3} \sin\left(\frac{\sqrt{3}}{2} t\right)\right);$$

7) $$\mathcal{F}(\cos 2x) = \int_{-\infty}^{+\infty} dx \, e^{-ikx} \cos 2x =$$

$$= \int_{-\infty}^{+\infty} dx \, e^{-ikx} \frac{e^{2ix} + e^{-2ix}}{2} = \frac{1}{2}\left(\int_{-\infty}^{+\infty} dx \, e^{i(2-k)x} + \right.$$

$$\left. + \int_{-\infty}^{+\infty} dx \, e^{-i(2+k)x} \right) = \frac{1}{2} \sqrt{2\pi} \left(\delta(2-k) + \delta(2+k) \right).$$

$$\mathcal{F}(\cos 2x) = \sqrt{\frac{\pi}{2}} \left(\delta(k+2) + \delta(k-2) \right).$$

8) $$F(y, y', x) = y'^2 - 2y^2 + y \cosh x$$

$$\frac{\partial F}{\partial y} = -4y \qquad \frac{\partial F}{\partial y'} = 2y'$$

$$2y'' + 4y - \cosh x = 0 \qquad y'' + 2y = \frac{1}{2} \cosh x$$

$$y(x) = A \cos(\sqrt{2} x) + B \sin(\sqrt{2} x) + y_p(x)$$

dove $y_p(x) =$ è soluzione particolare.