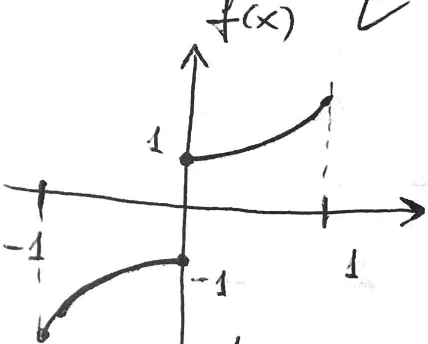
(1)

$$f(x) = \begin{cases} -e^{-x} & -1 \le x < 0 \\ e^{x} & 0 \le x < 1 \end{cases} \qquad T = 2 \qquad \omega = \frac{2\pi}{T} = \pi$$



f è dispari. $\quad a_k = 0$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} dx\, f(x) \sin k\omega x =$$

$$= \frac{4}{T}\int_{0}^{T/2} dx\, e^{x} \sin k\omega x = 2\int_0^1 dx\, e^x \sin(k\pi x) =$$

Notiamo che

$$\int dx\, e^x \sin k\pi x = -\frac{e^x \cos k\pi x}{k\pi} + \frac{1}{k\pi}\int dx\, e^x \cos k\pi x =$$

$$= -\frac{e^x \cos k\pi x}{k\pi} + \frac{1}{k\pi}\left(\frac{e^x \sin k\pi x}{k\pi} - \frac{1}{k\pi}\int dx\, e^x \sin k\pi x\right)$$

$$\Rightarrow \int dx\, e^x \sin k\pi x = \left(1 + \frac{1}{k^2\pi^2}\right)^{-1}\left[\frac{\sin k\pi x - k\pi \cos k\pi x}{k^2\pi^2}\right] e^x ;$$

$$\int_0^1 dx\, e^x \sin k\pi x = \left(\frac{k^2\pi^2+1}{k^2\pi^2}\right)^{-1}\left[e\left(\frac{-k\pi \cos k\pi}{k^2\pi^2}\right) - \frac{-k\pi}{k^2\pi^2}\right]$$

$$= \left(\frac{k^2\pi^2+1}{k^2\pi^2}\right)^{-1} \frac{e\, k\pi (-1)^{k+1} + k\pi}{k^2\pi^2}$$

$$= \frac{\cancel{k^2\pi^2}}{1+k^2\pi^2}\, \frac{k\pi\left(e(-1)^{k+1} + 1\right)}{\cancel{k^2\pi^2}}$$

$$b_k = 2\,\frac{k\pi}{1+k^2\pi^2}\left(1 + e(-1)^{k+1}\right) ;$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2n\pi}{1+n^2\pi^2}\left[1+e(-1)^{n+1}\right] \sin n\pi x$$

(2) $$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2 A_\rho\right) + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi}A_\phi$$

$$\cancel{\frac{\partial A_\rho}{\partial\rho} = 3\rho} \qquad \rho^2 A_\rho = 3\rho^4; \quad \sin\theta A_\theta = 2\rho\cos\theta\sin\theta$$

$$\frac{\partial A_\phi}{\partial\phi} = \rho\cos\theta\cos\phi;$$

$$\overline{\nabla}\cdot\overline{A} = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(3\rho^4\right) + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}\left(2\rho\cos\theta\sin\theta\right) +$$

$$+\frac{1}{\rho\sin\theta}\rho\cos\theta\cos\phi =$$

$$= \frac{1}{\cancel{\rho^2}}12\rho^{\cancel{3}} + \frac{1}{\cancel{\rho}\sin\theta}\left(-2\cancel{\rho}\sin^2\theta + 2\cancel{\rho}\cos^2\theta\right) + \frac{\cos\phi\sin\phi}{\sin\theta}$$

$$= 12\rho - 2\sin\theta + 2\frac{\cos^2\theta}{\sin\theta} + \frac{\cos\phi\sin\phi}{\sin\theta}$$

$$\vec{r} = (3\sqrt{2}, -3\sqrt{2}, 4)$$

$$\rho = \sqrt{18+18+16} = \sqrt{52} = 2\sqrt{13}$$

$$\theta = \arccos\left(\frac{4}{2\sqrt{13}}\right) = \arccos\frac{2\sqrt{13}}{13}$$

$$\varphi = \arcsin\left(\frac{-3\sqrt{2}}{6}\right) = -\arcsin\frac{\sqrt{2}}{2} = -\pi/4$$

$$\sqrt{18+18} = 6$$

$$\vec{\nabla}\cdot\vec{A} = 24\sqrt{13} - 2\sqrt{1-\left(\frac{2\sqrt{13}}{13}\right)^2} + 2\frac{\left(\frac{2\sqrt{13}}{13}\right)^2}{\sqrt{1-\left(\frac{2\sqrt{13}}{13}\right)^2}} + \frac{-\left(\frac{\sqrt{2}}{2}\right)^2}{\sqrt{1-\left(\frac{2\sqrt{13}}{13}\right)^2}} =$$

$$= 24\sqrt{13} - 2\sqrt{\frac{169-52}{169}} + 2\,\frac{\frac{52}{169}}{\sqrt{\frac{169-52}{169}}} - \frac{1/2}{\sqrt{\frac{169-52}{169}}}$$

$$= 24\sqrt{13} - \frac{2}{13}\sqrt{117} + 2\,\frac{52}{13\sqrt{117}} - \frac{1}{2}\,\frac{13}{\sqrt{117}} = \dots = \frac{613}{2\sqrt{13}}$$

③ $\Sigma(\theta, z) = (R\cos\theta, R\sin\theta, z)$

$\overline{\Sigma}_\theta = (-R\sin\theta, R\cos\theta, 0)$

$\overline{\Sigma}_z = (0, 0, 1)$

$$\overline{\Sigma}_\theta \times \overline{\Sigma}_z = \left(-R\sin\theta, R\cos\theta, 0\right) \times (0,0,1) =$$

$$= (R\cos\theta, R\sin\theta, 0)$$

$$\hat{n}(\theta, z) = \frac{\Sigma_\theta \times \Sigma_z}{|\Sigma_\theta \times \Sigma_z|} = (\cos\theta, \sin\theta, 0) \quad \left(\begin{matrix}\text{versore}\\ \text{superficie}\\ \text{laterale}\end{matrix}\right)$$

$$\left.\begin{matrix}\hat{n}_\uparrow = (0,0,1)\\ \hat{n}_\downarrow = (0,0,-1)\end{matrix}\right\} \text{ versori superficie di base}$$

$$\vec{F}(\vec{r}) = \left(-\frac{\cancel{\rho}\sin\theta}{\cancel{\rho^4}\,{}^3}, \frac{\cancel{\rho}\cos\theta}{\cancel{\rho^4}\,{}^3}, 1\right) = \left(-\frac{\sin\theta}{\rho^3}, \frac{\cos\theta}{\rho^3}, 1\right)$$

$$\Phi(\vec{F}) = \int_{\text{Sup. lat}} \vec{F}\cdot\hat{n}(\theta,z)\, R\,d\theta\,dz + \int_{\text{Sup. base}} \vec{F}\cdot\hat{n}_\uparrow\, \rho\,d\rho\,d\theta +$$

$$+ \int_{\text{Sup. base}} \vec{F}\cdot\hat{n}_\downarrow\, \rho\,d\rho\,d\theta = \iint_{\text{Sup. lat.}} \left(-\frac{\sin\theta}{R^3} R\cos\theta + \frac{\cos\theta}{R^3} R\sin\theta\right) R\,d\theta\,dz +$$

$$+\int_{Seg\uparrow} \cancel{1\,\rho\, d\rho\, d\theta} + \int_{Seg\downarrow} \cancel{(-1)\,\rho\, d\rho\, d\theta} =$$

$$= \frac{1}{R}\, h \int^{2\pi} \left(\cancel{-\sin\theta\cos\theta} + \cancel{\cos\theta\sin\theta}\right) d\theta = 0$$

Risultato scontato essendo il flusso in un campo "ingue-dico" nel piano xy ed un campo uniforme lungo z.

(4) $f(z) = \ln z = \ln(x+iy) = \ln(\rho\, e^{ik\theta}) =$

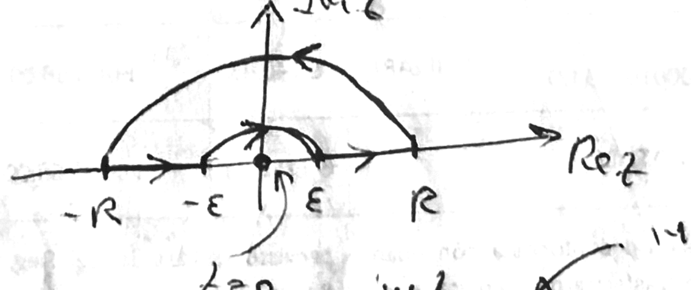
$= \ln\rho + ik\theta\,;$

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} \longrightarrow \frac{1}{x+iy} = \frac{1}{\cancel{i}}\,\frac{\cancel{i}}{x+iy} \qquad \underline{\text{ok}}$$

$$\frac{\partial f}{\partial \rho} = \frac{1}{i\rho}\frac{\partial f}{\partial \theta} \longrightarrow \frac{1}{\rho} = \frac{1}{\cancel{i}\rho}\,\cancel{i} \quad (\text{se } k=1) \qquad \underline{\text{ok}}$$

(5) $\displaystyle\int_{-\infty}^{+\infty} \frac{\sin mx}{x}\, dx \longrightarrow \oint \frac{e^{imz}}{z}\, dz$

$z=0$ (polo)



$$\oint \frac{e^{imz}}{z}\, dz = 2\pi i\, \text{Res}\left(\frac{e^{imz}}{z}, z_0\right) = 0$$

interno

la funzione è olomorfa nella parte di piano delimitata dal percorso.

$$\oint \frac{e^{imz}}{z}\, dx = \int_{-R}^{-\varepsilon} \frac{e^{imx}}{x}\, dx + \int_{\gamma_\varepsilon} \frac{e^{imz}}{z}\, dz + \int_{\varepsilon}^{R} \frac{e^{imx}}{x}\, dx +$$

$$+\int_{\Gamma_R} \frac{e^{imz}}{z}\,dz = 0$$

Nel $R \to \infty$ $\int_{\Gamma_R} \frac{e^{imz}}{z}\,dz \to 0$ (Teorema grande cerchio)

Nel $\varepsilon \to 0$ $\int_{\gamma_\varepsilon} \frac{e^{imz}}{z}\,dz \neq 0$ (non verifica teorema piccolo cerchio)

$$\int_{-R}^{-\varepsilon} \frac{e^{imx}}{x}\,dx + \int_{\varepsilon}^{R} \frac{e^{imx}}{x}\,dx = -\int_{\gamma_\varepsilon} \frac{e^{imz}}{z}\,dz$$

$$\gamma_\varepsilon \Rightarrow \varepsilon e^{i\theta} = z \quad dz = i\varepsilon e^{i\theta}\,d\theta = iz\,d\theta$$

$$\int_{\gamma_\varepsilon} \frac{e^{imz}}{z}\,dz \simeq \int_{\pi}^{0} \frac{1+im\varepsilon e^{i\theta}}{\cancel{z}}\, i\cancel{z}\,d\theta =$$

$$= i\int_{\pi}^{0} \left(1+im\varepsilon e^{i\theta}\right)d\theta \xrightarrow[\varepsilon \to 0]{} i\int_{\pi}^{0} d\theta = -i\pi$$

$$\int_{-R}^{-\varepsilon} \frac{e^{imx}}{x}\,dx + \int_{\varepsilon}^{R} \frac{e^{imx}}{x}\,dx = \int_{R}^{\varepsilon} \frac{e^{-imx}}{(-x)}(-dx) + \int_{\varepsilon}^{R} \frac{e^{imx}}{x}\,dx$$

$$= -\int_{\varepsilon}^{R} \frac{e^{-imx}}{x}\,dx + \int_{\varepsilon}^{R} \frac{e^{imx}}{x}\,dx = \int_{\varepsilon}^{R} \frac{e^{imx} - e^{-imx}}{x}\,dx =$$

$$= \int_{\varepsilon}^{R} \frac{2i\sin mx}{x}\,dx$$

$$2i\int_{\varepsilon}^{R} \frac{\sin x}{x}\,dx = -(-i\pi)$$

$R \to \infty$ e $\varepsilon \to 0$

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \pi/2 , \qquad \boxed{\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx = \pi}$$

(6) $$\begin{cases} y'' + y' = t^2 \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \Rightarrow s^2 \hat{y} + s\hat{y} = \frac{2}{s^3}$$

$$\hat{y} = \frac{2}{s^3}\,\frac{1}{s^2+s} = 2\,\frac{1}{s^4}\,\frac{1}{s+1}$$

$$\frac{1}{s+1} \to e^{-t} \qquad \frac{1}{s^4} \to t^3/6$$

$$y(t) = 2\int_0^t d\tau\, \frac{\tau^3}{6}\, e^{-(t-\tau)} = \frac{1}{3}\int_0^t d\tau\, \tau^3 e^{-(t-\tau)} =$$

$$= \dots = 2e^{-t} - 2 + 2t - t^2 + t^3/3 .$$

(7) $$f(x) = \sin x - \cos 2x$$

$$\hat{f}(k) = \int_{-\infty}^{+\infty} dx\, (\sin x - \cos 2x)\, e^{-ikx} =$$

$$= \int_{-\infty}^{+\infty} dx \left(\frac{e^{ix} - e^{-ix}}{2i} - \frac{e^{i2x} + e^{-i2x}}{2} \right) e^{-ikx} =$$

$$= \frac{1}{2i}\int_{-\infty}^{+\infty} dx\, e^{i(1-k)x} - \frac{1}{2i}\int_{-\infty}^{+\infty} dx\, e^{-i(1+k)x} +$$

$$- \frac{1}{2}\int_{-\infty}^{+\infty} dx\, e^{i(2-k)x} - \frac{1}{2}\int_{-\infty}^{+\infty} dx\, e^{-i(2+k)x} =$$

$$= \frac{1}{2i}\Big(\delta(1-k) - \delta(1+k)\Big) - \frac{1}{2}\Big(\delta(2-k) + \delta(2+k)\Big) .$$

(8) $$\delta\int_{-1}^{1}\left(y'^2 + \alpha y^2 - 2\cos x\right) dx = 0$$

$$\frac{d}{dx}\frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y} = 0 \qquad 2y'' - 2\alpha y = 0$$

$y'' - \alpha y = 0$ se $\alpha < 0$ $\quad y = A\cos\sqrt{\alpha}\,x + B\sin\sqrt{-\alpha}\,x$

se $\alpha > 0$ $\quad y = A\,e^{\sqrt{\alpha}\,x} + B\,e^{-\sqrt{\alpha}\,x}$